Serigne Touba Sall & Gane Samb Lo, Afrika Statistika, Vol.2, n°1, 2007, pp.44-58
The Asymptotic Theory of Intensity Poverty
in View of Extreme values Theory For Two Simples Cases

# THE ASYMPTOTIC THEORY OF THE POVERTY INTENSITY IN VIEW OF EXTREME VALUE THEORY FOR TWO SIMPLE CASES.

GANE SAMB LO AND SERIGNE TOUBA SALL

Abstract. Let $Y_1, Y_2$ .... be independent observations of the income variable of some given population, with underlying distribution G. Given a poverty line Z, then for each n$\geq$ 1, $q = q_n$ is the numbers of poor in some given the population. The general form of poverty measures used by economists to monitor the welfare evolution of this population is

$$P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^{q} c(n,q,j)\ d\left(\frac{Z - Y_{j,n}}{Z}\right).$$

This class includes the most popular poverty measures like the Sen, Shorrocks and Greer-Foster-Thorbecke statistics. We give a complete asymptotic normality theory in the frame of extreme value theory. In this paper, the two simple cases of the Pareto and exponential distributions are studied with the intensity poverty. Simulations and applications to the senegalese data are driven.

**Résumé**

Soit $Y_1, Y_2$ .... des observations indépendantes de la variable revenu d'une population donnée. Etant donné un seuil de pauvreté Z, alors $q_n$ est pour tout n$\geq$ 1 le nombre de pauvres dans la population. La forme générale des indicateurs de pauvreté est

$$P_n = \frac{1}{a(q)b(n)} \sum_{j=1}^{q} c(n,q,j)\ d\left(\frac{Z - Y_{j,n}}{Z}\right).$$

Cette classe de mesures contient les indicateurs classiques tels que ceux de Sen, de Shorrocks et de Greer-Foster-Thorbecke. Dans cet article, nous préparons la théorie asymtotique complète de ces indicateurs dans le cadre des valeurs extrêmes. Nous nous concentrons ici sur l'intensité de la pauvreté pour deux cas simples : La loi de Paréto et la loi exponentielle. Des simulations et des applications sur les données du Sénégal sont présentées.

## 1. INTRODUCTION.

Poverty is measured in Economics by statistics of the general form

$$P_N = \frac{1}{a(Q)b(N)} \sum_{j=1}^{Q} c(N,Q,j)\ d\left(\frac{Z - Y_{j,N}}{Z}\right) \tag{1.1}$$

where a, b, c, d are given functions, Q is the number of poor in the studied population P of size N, Z the poverty line and $Y_{1,N} \leq Y_{2,N} \leq ... \leq Y_{N,N}$ are

2000 *Mathematics Subject Classification.* 2E20,62E17, 62G30, 91B82, Secondary 60G15
JEL Classification: I32.
*Key words and phrases.* Poverty measurement and analysis, asymptotic normality, empirical processes, Gaussian approximations, estimation and simulations, extreme value theory.

Received on January 20,2006. Revised version accepted on May 10, 2007 

the ordered incomes of the individuals of P. The poverty line Z is defined by economic specialists or governmental authorities so that any individual or household with income (say yearly) less than Z is considered as poor. The economic welfare of the population $\mathcal{P}$ is monitored both by the measures (1.1) and other measures of the inequality of the income distribution. A

very large class of poverty measures can be found in the literature. One may divide them into two classes. The first includes the non weighted ones, for which c(N, Q, j)$\equiv$ 1. The most popular of them is the Foster-Greer-Thorbecke class which is, for $\alpha \geq 0$,

$$FGT(\alpha) = \frac{1}{N}\sum_{j=1}^{Q}\left(\frac{Z - Y_{j,N}}{Z}\right)^{\alpha}. \tag{1.2}$$

For $\alpha = 0$, (1.2) reduces to Q/N, the headcount of the poor while for $\alpha = 1$ and $\alpha = 2$, it is respectively interpreted as the severity of poverty and the depth in poverty. The second class consists of the weighted measures. We mention here two of its famous members, the Sen(1976) measure

$$P_{SE,N} = \frac{2}{N(Q+1)}\sum_{j=1}^{Q}(Q - j + 1)\left(\frac{Z - Y_{j,N}}{Z}\right) \tag{1.3}$$

and the Shorroks(1995) one

$$P_{SH,N} = \frac{1}{N^2}\sum_{j=1}^{Q}(2N - 2j + 1)\left(\frac{Z - Y_{j,N}}{Z}\right). \tag{1.4}$$

These discrete statistics have been widely prospected in the frame of poverty reduction. The interested readers are referred to [9], [10], [14], [18].

We now want to settle an asymptotic theory for the statistics (1.1). We suppose that we have independent and identically observations of the income $Y_1, Y_2$ .... with underlying distribution G with lower endpoint $y_0 = \inf\{x, G(x) > 0\} \geq 0$. Given a poverty line Z, we have the (random) number of poor $q_n = q$ in the sample of size n : $Y_1, Y_2, ..., Y_n$. The sample measure of poverty becomes

$$P_n = \frac{1}{a(q)b(n)}\sum_{j=1}^{q}c(n,q,j)\; d\left(\frac{Z - Y_{j,n}}{Z}\right). \tag{1.5}$$

Clearly, this depends on the q lower extreme values and suggests to use the extreme value theory to handle such statistics. If q/n$\rightarrow$ 0, the study will only need the lower tail of G. But for many poor countries q/n is greater than 50% so that the reasonable hypothesis must be q/n$\rightarrow \xi = G(Z) \in ]0,1[$. Thus, at most the lower tail and the center of G will be used. This is in conformity with the focalization hypothesis of poverty measure which says that any poverty statistics is strictly function of the poor income. In général, the law of such statistics are usually guided by the extremal domain of attraction of the sequence of minima $Y_{1,n}$, n$\geq$ 1. However, observe that much of the theory of extreme value is set up by using the extremal domain of attraction of the maxima. We make a brief recall of our needs in extreme value theory using maxima in Section 2 . In Section 3, we state our results for the intensity poverty whose the asymptotic theory is completely described for the Pareto and the exponential cases.

## 2. Basics of extreme values theory using Maxima

Let $Y_1, Y_2, ...$ be a sequence of independent and indentically distributed random variable with $P(Y_1 \leq x) = G(x)$ and G(0) = 0. The estimation of the law of the maximum $Y_{n,n} = \max(Y_1, ......., Y_n)$ is one of the most important questions of the extreme value theory. Recall that $Y_{n,n}$ converges in type to some random variable Z with non degenerated distribution function $H(.)$ if and only if there exists two sequences of real numbers $(a_n > 0, \ \ n \geq 1)$ and $(b_n, \ n \geq 1)$ such that

$$\lim_{n\to\infty} \mathbb{P}(Y_{n,n} \leq a_n x + b_n) = \lim_{n\to\infty} G^n(a_n x + b_n) = H(x), \quad \forall x \in \mathbb{R}. \tag{2.1}$$

It is then said that G is in the domain of attraction of H, which is denoted by G$\in D(H)$. It is well-known that H is necessarily one of these three types :

$$H(x) = \varphi_\alpha(x) = \exp(-x^{-\alpha})\ \mathbb{I}_{(x\geq 0)}\ ; \quad (type\ I) \tag{2.2}$$

for $\alpha > 0$,

$$H(x) = \Lambda(x) = \exp\left(-e^{-x}\right)\ ; \qquad (type\ II) \tag{2.3}$$

$$H(x) = \psi_\gamma(x) = \exp(-(-x)^\gamma)\ \mathbb{I}_{(x\leq 0)} + \mathbb{I}_{(x>0)}\ ;\ \gamma > 0\ \ ;\ \ (type\ III) \tag{2.4}$$

where $I_A$ stands for the indicator function of the set $A$. It should be noticed that the function H has to be seen as an equivalence :

$$H_1 \mathcal{R}\ H_2\ \Leftrightarrow (\exists\ (a, b) \in \mathbb{R}_+^* \times \mathbb{R}, \forall\ x \in \mathbb{R},\ H_1(x) = H_2(ax + b)) \tag{2.5}$$

In this case, any of the function H in (2.2), (2.3) and (2.4) may be represented as

$$H(x) = H_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}),\ \text{for } 1{+}\gamma x > 0,$$

where $(1 + \gamma x)^{-1/\gamma}$ is interpreted as $e^{-x}$ for $\gamma = 0$. The values $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ are respectively related to the types I, II et III.

The class of the whole domain of attraction (the set of all distribution functions attracted to some H) is denoted by $\mathfrak{D} = \cup_{\alpha>0} \cup_{\gamma>0} D(\varphi_\alpha) \cup D(\psi_\gamma) \cup D(\Lambda)$.

The interested reader is referred to [5], [7] or [15] for a general introduction to extreme value theory. The characterization of these three domains received very great attention in the two past decades. The frame of extreme value theory has also been widely used to find the asymptotic normality of sums of extreme values as well as Hill type statistics (see [12], mainly and [13], [16], [6], [11]).

The following representations are then used for the generalized inverse of G defined by $G^{\leftarrow}(s) = \inf\{x, G(x) \geq s\}$, $0 \leq s \leq 1$. First, w have for G$\in D(\varphi_\gamma)$, $z_0 = \sup\{x, G(x) < 1\} = +\infty$ and

$$G^{\leftarrow}(1-u) = c(1+f(u))\ u^{1/\gamma} \exp(\int_u^1 b(t)t^{-1}dt), \quad 0 < u < 1, \tag{2.6}$$

Next, for G$\in D(\psi_\gamma)$, $z_0 = \sup\{x, G(x) < 1\} < +\infty$ and

$$z_0 - G^{\leftarrow}(1-u) = c(1+f(u))\ u^{1/\gamma} \exp(\int_u^1 b(t)t^{-1}dt), 0 < u < 1 \tag{2.7}$$

and, finally, for G$\in D(\Lambda)$, $z_0 = \sup\{x, G(x) < 1\} \leq +\infty$.

$$G^{\leftarrow}(1-u) = d - s(u) + \int_u^1 s(t)t^{-1}dt), 0 < u < 1, \tag{2.8}$$

where $s(u) = c(1+f(u))\exp(\int_u^1 b(t)t^{-1}dt)$, $0 < u < 1$. In each of these formulae, f and b are functions such taht (f(u), b(u))→ (0,0) when u→ 0, while c> 0 and d are constants. (2.6) and (2.7) are the Karamata representations while (2.8) is the one of de Haan.

To finish, we recall the characterization on G for the different domains. First, by Theorem 2.4.3 in [7], G$\in D(\Lambda)$ iff

$$\frac{1-G(t+xr(t))}{1-G(t)} \to e^{-x}, \text{ as } x \to z_0 \tag{2.9}$$

for any x and for some positive function r(·). Moreover, all the functions r(·) in (2.9) are equivalent between them as x→ $z_0$ Further, by Theorem 2.4.1 in [6], G $\in D(\Lambda)$ iff

$$\forall x > 0, \{G^{\leftarrow}(1-ux) - G^{\leftarrow}(1-u)\}/r(u) \quad \to \text{ -}\log x, \tag{2.10}$$

for some positive function r(·). Notice that $z_0$ may be finite or not. Next, G$\in D(\varphi_\gamma)$ iff $z_0 = +\infty$ and

$$\forall \lambda > 0,\ (1-G(\lambda x))/(1-G(x)) \to \lambda^{-\gamma}. \tag{2.11}$$

In this case, G admits the Karamata representation

$$1 - G(x) = c(1+f_1(x))\ x^{-\gamma} \exp(\int_1^x b_1(t)t^{-1}dt), \quad y_0 < x < z_0, \tag{2.12}$$

where $(f_1(u), b_1(u)) \to (0,0)$ as u→ 0. The function

$$L(x) = c(1+f_1(u))\exp(\int_1^x b_1(t)t^{-1}dt)$$

is the representation of a function slowly varying at infinity that is

$$L(\lambda x)/L(x) \to 1 \ as\ x \uparrow \infty$$

for any $\lambda > 1$. Finally G$\in D(\psi_\gamma)$ iff $z_0 < \infty$ and $G(z_0 - 1/\cdot) \in D(\varphi_\gamma)$ .

Now, we must remark in our frame that Y is an income variable. Then its lower endpoint $y_0$ is not negative. This allows us to study (1.5) via the transform $X=1/(Y-y_0)$. Throughout this paper, the distribution function of