Asymptotic behavior for the extreme values of a regression model

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Abstract

We consider a class of linear regression model Y_t with (ζ_t) a white noise error process. We assume that the explanatory variable (X_t) and (ζ_t) are independent. We show by means of a point process technique that the asymptotic distribution of $\max_{1\leq k\leq n} Y_k$ is the same as the one of $\max_{1\leq k\leq n} X_k$ under specific conditions on the noise process. The conditions say that the tail of (ζ_t) is lighter that the tail of (X_t).

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1 Introduction

In this paper we are interested to study the asymptotic behavior of the maxima for a regression model defined by the following scheme :

$$Y_t = aX_t + b + \zeta_t \tag{1}$$

where (ζ_t) is a white noise process and (X_t) a sequence of independent and identically distributed random variables (iid rvs) independent of (ζ_t). We assume that the common df F of X_1 belongs to the extreme value domain, see Gnedenko (1943).

First, let us recall the following result which is the basis of classical extreme value theory.

Theorem 1 *(Fisher-Tippet theorem, limit laws for maxima)*
Let (X_n) be a sequence of iid rvs. If there exist two sequences $(a_n > 0)$, (b_n) and some non-degenerate df H such that

$$a_n^{-1}(\max_{1\le k\le n} X_k - b_n) \xrightarrow{d} H, \tag{2}$$

then H belongs to the type of one of the following three dfs :

$$\begin{array}{llcll} \textit{Gumbel} & \Lambda(x) & = & \exp(-e^{-x}), & x \in \mathbb{R}, \\ \textit{Fréchet} & \phi_\alpha(x) & = & \exp(-x^{-\alpha}), & x > 0,\ \alpha > 0, \\ \textit{Weibull} & \psi_\alpha(x) & = & \exp(-(-x)^\alpha), & x \le 0,\ \alpha > 0. \quad \blacksquare \end{array}$$

From now on we refer to the centring constants a_n and the normalizing constants b_n jointly as norming constants. We say that the df F of X belongs to the maximum domain of attraction of the extreme value distribution H if (2) holds. We write $F \in D(H)$ $(X \in D(H))$.

The paper is organized as follows. In Section 2 we present the model. In Section 3, we study the extreme value behavior of the explained variable

(Y_t) by means of a point process technique. Under condition on the noise process, we show that the asymptotic distribution of $\max\limits_{1\le k\le n} Y_k$ is a Fréchet distribution when the distribution function F of the explanatory variable belongs to the Féchet domain of attraction. The condition says that the tail of the noise process is relatively light compared to that of the explanatory variable. Some examples are given. In Section 4, we give equivalent result when the distribution function F of the explanatory variable belongs to the Gumbel domain of attraction. Section 5 is devoted to the conclusion.

2 The model

We consider the regression model defined in (1) where (X_t) is a sequence of iid rvs independent of the sequence (ζ_t). We assume that $F \in D(\Lambda) \cup D(\phi_\alpha)$, $\alpha > 0$. We suppose that the distribution of (ζ_t) belongs to a large class of distribution (not necessarily gaussian). The conditions stated in the main theorem says that the tail of (ζ_t) is lighter that the tail of (X_t). We show that the asymptotic distribution of the extreme values of the variable Y is the same as the one of the variable X if $a > 0$ in (1).

We introduce a proposition which allows us to deal with a model simpler than (1) using a linear transformation

Proposition 1 *Let (V_i) be a sequence of iid rvs, $(c_n > 0)$ and (d_n) two sequences of $\mathbb{R}$ such that for all continuity point x of H,*

$$\mathbb{P}\{\max_{1\le k\le n} V_k \le c_n x + d_n\} \longrightarrow H(x),$$

where H is a non degenerated df. If f is an increasing linear function and if $V_i' = f(V_i)$ then with $c_n' = c_n f'(d_n)$ and $d_n' = f(d_n)$, we have

$$\mathbb{P}\{\max_{1\le k\le n} V_k' \le c_n' x + d_n'\} \longrightarrow H(x).$$

$\blacksquare$

Let us now consider the linear transformation : $X'_t = aX_t + b$ where $a > 0$. The proposition shows that $\max\limits_{1\le k\le n} X_k$ and $\max\limits_{1\le k\le n} X'_k$ have the same asymptotic behaviour. If X belongs to $D(H)$ with the norming constants a_n and b_n then using the previous result, we get

$$\mathbb{P}\{\max_{1\le k\le n} X'_k \le a'_n x + b'_n\} \longrightarrow H(x),$$

with $a'_n = aa_n$ and $b'_n = ab_n + b$: (thus it suffices to put $a = 1$ and $b = 0$ in (1). From now on, we consider the process (Y_t) defined by the following scheme

$$Y_t = X_t + \zeta_t \tag{3}$$

where (X_t) is a sequence of iid rvs with distribution function F and (ζ_t) is a white noise process independent of (X_t).

3 Fréchet domain

Assume now that the distribution function F which characterizes the r.v. X of the model (6) belongs to the Frechet domain, i.e., $F \in D(\phi_\alpha)$ with $\alpha > 0$. Denote

$$a_n = F^{-1}(1 - \frac{1}{n}) = n^{\frac{1}{\alpha}} L(n), \qquad \text{and} \qquad b_n = 0$$

where L is a slowly varying fonction at ∞, i.e. $\lim_{x\to+\infty} \frac{L(tx)}{L(x)} = 1,\ t > 0$, F^{-1} the generalized inverse function of F defined by

$$F^{-1}(y) = \inf\{x \in \mathbb{R}, F(x) \ge y\}.$$

We also recall the following tail balancing condition for a stationary process (ζ_t) given in Davis and Resnick (1985)

$$\lim_{x\to+\infty} \frac{\mathbb{P}\{\zeta_1 > x\}}{\mathbb{P}\{|\zeta_1| > x\}} = \pi_0, \qquad \lim_{x\to+\infty} \frac{\mathbb{P}\{\zeta_1 < -x\}}{\mathbb{P}\{|\zeta_1| > x\}} = 1 - \pi_0, \tag{4}$$

where $0 < \pi_0 \leq 1$.

Let $\mathcal{E}$ be the Borel σ-field of subsets of a set $E \subset \mathrm{I\!R}^k$. For $x \in E$ and $A \in \mathcal{E}$, we define the measure ε_x on $\mathcal{E}$ by

$$\varepsilon_x(A) = \begin{cases} 1 \quad , & x \in A \\ 0 \quad , & x \notin A. \end{cases}$$

Let $\{x_i, i \geq 1\}$ be a a countable collection (not necessary distinct) of point of the space E. A point measure m is defined to be a measure of the form $m = \sum_{k=1}^{\infty} \varepsilon_{x_i}$ which is nonnegative integer valued and finite on relatively compact subsets of E. The class of point measures is denoted by $\mathcal{M}_p(E)$. Let also μ be a Radon measure on $\mathcal{E}$, a Poisson random measure with mean measure μ will be denoted by $PRM(\mu)$.

Now we introduce :

$$N_n = \sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} Y_k)},$$

and

$$N_n^{(1)} = \sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} X_k)}, \qquad N_n^2 = \sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} \zeta_k)},$$

Thus, N_n, $N_n^{(1)}$ and $N_n^{(2)}$ are PRM. The two next lemma give the convergence of these processes. Part 1) of lemma 1 is due to Resnick (1987).

Lemma 1 *1) Let (X_t) be a sequence of iid rvs with common distribution F belonging to $D(\phi_\alpha)$ with $\alpha > 0$. Suppose $F(0) = 0$ so that $X_i > 0$ a.s. Then*

$$N_n^{(1)} \xrightarrow{d} N_1 \qquad as \qquad n \to +\infty,$$

in $\mathcal{M}_p([0,\ \infty) \times (0,\ \infty])$, where N_1 is a PRM($\lambda \times \nu_1$) with λ the Lebesgue measure on $[0,\ \infty)$ and $\nu_1(x,\ \infty] = x^{-\alpha},\ x > 0$.

2.) Suppose that the sequence of iid rvs (ζ_t) satisties the tail balancing condition specified in (4) and the following condition fot $x > 0$

$$n\bar{F}_{|\zeta|}(a_n x) \to 0, \qquad as \qquad n \to +\infty. \tag{5}$$

Then

$$N_n^{(2)} \xrightarrow{d} N_2 \equiv 0 \qquad as \qquad n \to +\infty,$$

in $\mathcal{M}_p([0,\ \infty) \times (-\infty,\ +\infty] \setminus \{0\})$. ∎

Proof : It suffices to show that

$$n\mathbb{P}\{a_n^{-1}\zeta_1 \in .\} \xrightarrow{\nu} 0 \qquad \text{as} \qquad n \to +\infty, \tag{6}$$

where $\xrightarrow{\nu}$ denotes vague convergence of measures.

First, using the tail balancing condition (4), we have for all $x < 0$,

$$\lim_{n\to+\infty} n\mathbb{P}\{a_n^{-1}\zeta_1 < x\} = \lim_{n\to+\infty} (1-\pi_0) n\mathbb{P}\{a_n^{-1}|\zeta_1| > -x\} \tag{7}$$

Morever, for all $x > 0$,

$$\lim_{n\to+\infty} n\mathbb{P}\{a_n^{-1}\zeta_1 > x\} = \lim_{n\to+\infty} \pi_0 n\mathbb{P}\{a_n^{-1}|\zeta_1| > x\}. \tag{8}$$

Using the assumption (5), the expressions (7) and (8), we establish (6) as claimed. ∎

The following lemma permits to get the convergence of the process N_n.

Lemma 2 *Assume that the process* (X_t) *and* (ζ_t) *satisfy the hypotheses of lemma 1 and* (Y_t) *verifies (3) then in the space* $\mathcal{M}_p([0,\ \infty) \times (-\infty,\ +\infty] \setminus \{0\})$

$$N_n \xrightarrow{d} N_1 + N_2 \qquad as \qquad n \to +\infty,$$

Proof : Here we adapt the proof of proposition 4.21 of Resnick (1987) applied to linear processes. We must show

$$d\Big(\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\ a_n^{-1}Z_k)},\ \ \sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\ a_n^{-1}X_k e_1)} + \varepsilon_{(\frac{k}{n},\ a_n^{-1}\zeta_k e_2)}\Big) \xrightarrow{P} 0 \tag{9}$$

where $Z_k = (X_k,\ \zeta_k)$, $e_1 = (1,\ 0)$ and $e_2 = (0,\ 1)$.

Where d is the vague metric on $\mathcal{M}_p([0,\ \infty) \times (-\infty,\ +\infty] \setminus \{0\})$. It suffices to check for all $f \in C_K^+([0,\ \infty) \times (-\infty,\ +\infty] \setminus \{0\})$ with support contained in $[0,\ 1] \times \{(x_1,\ x_2));\ |x_1| > \delta$ or $|x_2| > \delta\}$, for some $\delta > 0$, that

$$I_n(f) - I_n^*(f) \xrightarrow{P} 0, \tag{10}$$

where I_n and I_n^* denote respectively the two terms of (9). Set

$$H = \{(x_1,\ x_2)) \in [-\infty,\ +\infty] :\ \ |x_1| > \delta \text{ and } \ |x_2| > \delta\}.$$

Then

$$I_n(f) = \int f dI_n = \int_{[0,\ 1]\times H^c} f dI_n + \int_{[0,\ 1]\times H} f dI_n.$$

Since

$$\begin{aligned} E[I_n([0,\ 1] \times H)] &= nP\{a_n^{-1} Z_k \in H\} \\ &\leq n\mathbb{P}\{a_n^{-1}|X_k| > \delta\} P\{a_n^{-1}|\zeta_k| > \delta\} \\ &\leq n\mathbb{P}\{a_n^{-1} X_k > \delta\} P\{a_n^{-1}|\zeta_1| > \delta\}. \end{aligned}$$

We have :

$$E[I_n([0,\ 1] \times H)] \to 0 \qquad \text{as} \qquad n \to +\infty,$$

and this readily implies,

$$I_n(f) = \int f dI_n = \int_{[0,\ 1]\times H^c} f dI_n + o_p(1).$$

Morever, we have obviously

$$I_n^*(f) = \int_{[0,\ 1]\times H^c} f dI_n^*.$$

To establish (10), we must show that the following expression

$$\sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1} Z_k) 1_{\{a_n^{-1} X_k \leq \delta,\ a_n^{-1}|\zeta_k| > \delta\}} - \sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1} \zeta_k e_2) 1_{\{a_n^{-1}\zeta_k > \delta\}} \tag{11}$$

$$+ \sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1} Z_k) 1_{\{a_n^{-1} X_k > \delta,\ a_n^{-1}|\zeta_k| \leq \delta\}} - \sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1} X_k e_1) 1_{\{a_n^{-1}|X_k| > \delta\}}$$

tends to 0 in probability. The first term of (11), which we denote by J_1 can be written in the following form

$$\sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1}Z_k)1_{\{a_n^{-1}X_k\le\delta,\ a_n^{-1}|\zeta_k|>\delta\}} \quad - \quad \sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1}\zeta_k e_2)1_{\{a_n^{-1}X_k\le\delta,\ a_n^{-1}|\zeta_k|>\delta\}}$$
$$- \quad \sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1}\zeta_k e_2)1_{\{a_n^{-1}X_k>\delta,\ a_n^{-1}|\zeta_k|>\delta\}}.$$

We have :

$$|J_1| \le \sum_{k=1}^{n} |f(\frac{k}{n},\ a_n^{-1}Z_k) - f(\frac{k}{n},\ a_n^{-1}\zeta_k e_2)|1_{\{a_n^{-1}X_k\le\delta,\ a_n^{-1}|\zeta_k|>\delta\}} \tag{12}$$
$$+\sum_{k=1}^{n} f(\frac{k}{n},\ a_n^{-1}\zeta_k e_2)1_{\{a_n^{-1}X_k>\delta,\ a_n^{-1}|\zeta_k|>\delta\}}.$$

Let denoye by A and B, the two terms of (12).

$$E(B) \le nP\{a_n^{-1}|X_k| > \delta\}\mathbb{P}\{a_n^{-1}|\zeta_k| > \delta\}\sup f(x).$$

Then

$$E(B) \to 0 \qquad \text{as} \qquad n \to +\infty.$$

The indicator function associated with A is bounded by $(0 < \eta < \delta)$

$$1_{\{a_n^{-1}X_k<\eta,\ a_n^{-1}|\zeta_k|>\delta\}} + 1_{\{a_n^{-1}X_k>\eta,\ a_n^{-1}|\zeta_k|>\eta\}}.$$

Therefore

$$E(A) \le \sup\{|f(s,\ x)) - f(s,\ x_2e_2)| : \quad |x_1| \le \eta,\ \ |x_2| > \delta\}n\mathbb{P}\{a_n^{-1}\zeta_k > \delta\}$$
$$+(Constante)n\mathbb{P}\{a_n^{-1}X_k > \eta\}P\{a_n^{-1}|\zeta_k| > \eta\}.$$

Since f is uniformly continuous on its compact support, the sup can be made as small as we like by choosing η small. By (5), the bound of $E(A)$ converges as $n \to +\infty$ to 0 and hence J_1 tends to 0 in probability. By similar arguments, the second term of (11) which we denote by J_2 tends to 0 in probability and the formulae (9) is proved. We have proved that

$$\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\ a_n^{-1}Z_k)} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\ a_n^{-1}X_k e_1)} + \varepsilon_{(\frac{k}{n},\ a_n^{-1}\zeta_k e_2)}$$

have the same weak behaviour.

Since

$$\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} X_k)} \xrightarrow{d} \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, j_k)},$$

and

$$\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} \zeta_k)} \xrightarrow{d} \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, l_k)},$$

we have by the continuous mapping theorem

$$\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} X_k e_1)} \xrightarrow{d} \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, j_k e_1)},$$

and

$$\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} X_k e_2)} \xrightarrow{d} \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, l_k e_2)}.$$

Which implies

$$I_n^* \xrightarrow{d} \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, j_k e_1)} + \varepsilon_{(t_k,\, l_k e_2)}.$$

An application of the continuous mapping theorem yield that :

$$\begin{aligned}
\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} Y_k)} &= T_1\Big(\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} Z_k)}\Big) \\
&\approx T_1\Big(\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} X_k e_1)} + \sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1} \zeta_k e_2)}\Big) \\
&\xrightarrow{d} \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, j_k e_1)} + \varepsilon_{(t_k,\, l_k e_2)} \\
&= \sum_{k=1}^{\infty} \varepsilon_{(t_k,\, j_k)} + \varepsilon_{(t_k,\, l_k)}.
\end{aligned}$$

To finish the proof, we notice that N_1 and N_2 are independent which is a consequence of the independence of X_t and ζ_t.

Using the Laplace functional, we get

$$\psi_{N_1+N_2}(g) = \exp - \int_{[0,\,\infty)\times(-\infty,\,+\infty)\setminus} (1 - e^{-g(x,\,y)}) dm(x,\ y)$$

where $m = \lambda \times \nu_1$, λ is the Lebesgue measure. This is the desired conclusion. ■

Now we give the main result when F belongs to the Fréchet domain.

Theorem 2 *Under the hypotheses of lemma 2, we have*

$$\mathbb{P}\{a_n^{-1}M_n \leq x\} \to \phi_\alpha(x), \;\; x > 0, \;\; \alpha > 0 \;\; \textit{as} \qquad n \to +\infty$$

where $M_n = \max_{1\leq k\leq n} Y_k$ *and* (Y_t) *verifies (3).*

Proof : Consider the mapping T_2 defined by

$$T_2(\sum_{k=1}^{\infty} \varepsilon_{(t_k,\, j_k)}) = \sup\{j_k,\ t_k \leq .\}.$$

Set

$$Y_n(t) = \begin{cases} a_n^{-1}M_{[nt]} & \text{if} \quad t \leq \frac{1}{n}, \qquad n \geq 1 \\ a_n^{-1}Y_1 & \text{if} \quad 0 < t \leq \frac{1}{n}, \end{cases}$$

T_2 is an a.s. continuous mapping from $\mathcal{M}_p([0,\ \infty \times (0,\ \infty])$. This relation and the continuous mapping theorem yield that

$$T_2(\sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\, a_n^{-1}Y_k)}) \stackrel{d}{=} Y_n(.) \to T_2(\sum_{k=1}^{\infty} \varepsilon_{(t_k,\, j_k)} + \varepsilon_{(t_k,\, j_k)}).$$

Denote by $Y(.)$ the extremal process limit. By the lemma 1 and 2, we obtain

$$\begin{aligned} \mathbb{P}\{Y(t) \leq x\} &= P\{N_1 + N_2([0,\ t) \times [x,\ \infty)) = 0\} \\ &= \exp\{-\lambda \times \nu_1([0,\ t) \times [x,\ \infty))\} \\ &= \exp\{-tx^{-\alpha}\}, \qquad x > 0.\blacksquare \end{aligned}$$

Theorem 1 permits to show that the asymptotic behaviour for the extreme value of (Y_t) defined in (3) belongs also to $D(\phi_\alpha)$. Now if we consider the norming constants $\alpha_n = aa_n$ and $b_n = ab_n + b$ the result is always true for the process (Y_t) defined in (1) using proposition 1.

Remarks and Examples :

The assumption (5) is verified by several classical distribution for ζ_t.

- Noise with Weibull-like distribution.

  We suppose here that the random variable ζ_t has a Weibull-like distribution

  $$\bar{F}_\zeta(x) = \exp(-cx^\tau l(x))$$

  where c and τ are positive constants and $\lim_{x \to +\infty} l(x) = 1$. Using the fact that $\dfrac{\log L(n)}{\log n} \to 0$ as $n \to +\infty$ (see Proposition 0.8 i) of Resnick (1987)), it is easy to see that

  $$n\bar{F}_{|\zeta|}(a_n x) = \exp(-c(a_n x)^\tau l(a_n x))$$

  goes to 0 as $n \to +\infty$.

- Generalized error distribution (GED) noise

  Suppose that the random variable ζ_t follows a GED distribution. The GED density is defined by

  $$f_\zeta(x) = c_0 exp(-k|x|^\gamma), \tag{13}$$

  with $c_0 > 0$, $\gamma > 0$ and $k > 0$. The expression (13) can also be written :

  $$f_\zeta(x) = \frac{\gamma \exp(-\frac{1}{2}|\frac{x}{\tau}|^\gamma)}{\tau 2^{1+\frac{1}{\gamma}} \Gamma(\frac{1}{\gamma})}, \qquad \gamma > 0$$

  with

  $$\tau = [\frac{2^{-\frac{2}{\gamma}} \Gamma(\frac{1}{\gamma})}{\Gamma(\frac{3}{\gamma})}]^{\frac{1}{2}}.$$

  This class of density contains the normal density ($\gamma = 2$) and the Laplace density ($\gamma = 1$), and has the uniform density as a limit ($\gamma \to +\infty$). It was firstly introduced by Subbotin (1923) as the exponential power distribution. The tail behavior of the noise process $(\zeta_t)_t$

which is characterized by such a density depends on the tail-thickness parameter γ. For instance, if $\gamma = 2$, then $\zeta_t \sim \mathcal{N}(0,1)$, while for $\gamma < 2$ the distribution has thicker tails than the Gaussian distribution. Denote by Φ_ζ the corresponding distribution function, we have

$$1 - \Phi_\zeta(x) \leq \frac{c_0}{\gamma k} x^{1-\gamma} e^{-kx^\gamma}, \qquad \gamma \geq 1/2.$$

Using again the fact that $\dfrac{\log L(n)}{\log n} \to 0$ as $n \to +\infty$, we establish the desired result.

- Noise with Pareto-like distribution

  If F_ζ is Pareto-like distribution function with parameter $\beta < \alpha$, then the condition (5) is not satisfied. If $\beta \geq \alpha$ then the tail of the error distribution is relatively light compared to that of X and then the condition (5) is satisfied.

4 Gumbel domain

Assume now that the distribution distribution F of X_t belongs to the Gumbel domain, i.e. $F \in D(\Lambda)$ with the norming coefficients a_n and b_n. Here we precise two technical lemma before giving the main result.

Lemma 3 *Let (X_t) be a sequence of iid rvs with common distribution function $F \in D(\Lambda)$ and (ζ_t) a sequence of iid rvs with marginal density f_ζ. We assume that (X_t) and (ζ_t) are independent $a_n \to \gamma^{-1} \in (0,\ \infty)$. Then in the space $\mathcal{M}_p([0,\ \infty) \times (-\infty,\ +\infty]^2)$*

$$N_n^{(3)} = \sum_{k=1}^{\infty} \varepsilon_{\left(\frac{k}{n},\ a_n^{-1}(X_k - b_n),\ a_n^{-1}\zeta_k\right)} \xrightarrow{d} N_3 \tag{15}$$

where N_3 is a PRM($\lambda \times \mu_3$), N_3 can be written in the form $\sum_{k=1}^{\infty} \varepsilon_{(t_k,\ U_k,\ V_k)}$, λ is Lebesgue measure on $[0,\ \infty)$ and $\mu_3(dx,\ dy) = e^{-x}dx\gamma^{-1}f_\zeta(\gamma^{-1}y)dy$.

Proof : In view of proposition 3.21 of Resnick (1987), it suffices to check for $(x,\ y) \in (-\infty,\ +\infty]^2$ that

$$n\mathbb{P}\{(a_n^{-1}(X_1 - b_n),\ a_n^{-1}\zeta_1) \in [x,\ +\infty] \times [y,\ +\infty]\} \to e^{-x}\bar{F}_{\zeta_1}(\gamma^{-1}y).$$

Since (ζ_t) and (X_t) are independent and $a_n \to \gamma^{-1}$, it is now straightforward to obtain the desired result.

Now we establish a convergence result for a point process based on the explained variable (Y_t).

Lemma 4 *Assume the same hypotheses than in lemma 3 and* $a_n \to \gamma^{-1} \in (0,\ \infty)$, *assume also that* (Y_t) *verifies (2) and*

$$\theta := \int_{\mathbb{R}} f_\zeta(t)e^{\gamma t}dt < +\infty. \tag{16}$$

Then in the space $\mathcal{M}_p([0,\ \infty) \times (-\infty,\ +\infty])$, *we get*

$$N_n^{(4)} = \sum_{k=1}^{\infty} \varepsilon_{(\frac{k}{n},\ a_n^{-1}(Y_k - b_n))} \xrightarrow{d} N_4 = \sum_{k=1}^{\infty} \varepsilon_{(t_k,\ U_k + V_k)}, \tag{17}$$

where N_4 *is a PRM*($\lambda \times \mu_4$), λ *is Lebesgue measure on* $[0,\ \infty)$ *and* $\mu_4(dx) = \theta e^{-x}dx$.

Proof : An application of the continuous mapping theorem to (15) gives (17). The computation of the mean measure $\lambda \times \mu_4$ needs (16). As a matter of fact, define the function T from $[0,\ \infty) \times (-\infty,\ +\infty]^2$ into $[0,\ \infty) \times (-\infty,\ +\infty]$ by

$$T(t,\ x,\ y) = \begin{cases} (t,\ (x+y)) & \text{if} \quad (x,\ y) \in \mathbb{R}^2 \\ (t,\ 0) & \text{if} \quad x = +\infty \ \text{ or } \ y = +\infty. \end{cases}$$

By the proposition 2.2 of Davis and Resnick (1988), the mapping

$$\hat{T}:\ \mathcal{M}_p([0,\ \infty) \times (-\infty,\ +\infty]^2) \to \mathcal{M}_p([0,\ \infty) \times (-\infty,\ +\infty])$$

defined by

$$\hat{T}(\sum_i \varepsilon_{x_i}) = \sum_i \varepsilon_{Tx_i}$$

is continuous. Then

$$N_n^{(4)} = \hat{T}(N_n^{(3)}) \xrightarrow{d} N_4 = \hat{T}(N_3)$$

where $\hat{T}(N_3)$ is a PRM with mean measure $\lambda \times \mu_4$ given by easy computation. Indeed

$$\begin{aligned}
\mu_4(z,\ +\infty] &= \mu_3 \circ T^{-1}(z,\ +\infty], \\
&= \mu_3\{(x,\ y),\ x+y>z\}, \\
&= \int_{\{x+y>z\}} e^{-x}\gamma^{-1} f_\zeta(\gamma^{-1}y)dxdy, \\
&= \int_{\mathbb{R}} \gamma^{-1} f_\zeta(\gamma^{-1}y) \int_{z-y}^{+\infty} e^{-x}dxdy, \\
&= \int_{\mathbb{R}} \gamma^{-1} f_\zeta(\gamma^{-1}y) e^{-z+y} dy, \\
&= e^{-z} \int_{\mathbb{R}} \gamma^{-1} f_\zeta(\gamma^{-1}y) e^{y} dy
\end{aligned}$$

Since the assumption (16) is verified, the result follows upon changing variable.

Theorem 3 *Under the hypotheses of lemma 4, we have*

$$\lim_{n\to+\infty} \mathbb{P}\{a_n^{-1}(\max_{1\le k\le n} Y_k - b_n) \le x\} = \Lambda^\theta(x), \ \ \textit{for all} \ \ x \in \mathbb{R}$$

where $\Lambda^\theta(x) = \exp(-\theta e^{-x})$.

Proof : The proof is identical to that of theorem 2 where $N_1 + N_2$ is replaced by the process limit N_4 defined in lemma 4. ■

By the same way as in 2.1, we can extend these results to the process (Y_t) defined in (1). The condition (16) is satisfied when the tail of the noise distribution is relatively light compared to that of X when the distribution of X belongs to the Gumbel domain of attraction. For instance (16) is satisfied by distribution function with bounded support, Gaussian distribution and the exponential distribution with parameter $\lambda \geq \gamma$.

5 Conclusion

In this paper the asymptotic behavior for the extreme values of a regression model is studied. Under conditions satisfied by several classical distributions, we have shown that the extreme value theory of the explained variable is the same as the one of the extreme value value of the explanatory variable. So statistical inference of extreme values about (Y_t) could be handled using the process (X_t). Applications of these results are provided in a fourthcoming paper. The same problem could be considered in the case of dependent sequences.

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