THE GENERALIZED POVERTY INDEX

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Abstract. We introduce the General Poverty Index (GPI), which summarizes most of the known and availbale poverty indices, in the form

$$GPI = \delta(\frac{A(Q,N,Z)}{NB(Q,N)}\sum_{j=1}^{Q} w(\mu_1 N + \mu_2 Q - \mu_3 j + \mu_4) d\left(\frac{Z - Y_{j,N}}{Z}\right)),$$

where

$$B(Q,N) = \sum_{j=1}^{Q} w(j)$$

$A(\cdot)$, $w(\cdot)$, *and* $d(\cdot)$ are given measurable functions, Q is the number of the poor in the population $\mathcal{P}$ of size N, Z is the poverty line and $Y_{1,N} \leq Y_{2,N} \leq ... \leq Y_{N,N}$ are the ordered incomes or expenses of the individuals or households of $\mathcal{P}$. We show here how the available indices based on the poverty gaps are derived from it. The aysmptotic normality in then established and particularized for the usual poverty measures for immediate applications to poor countries data.

1. Introduction

Let $Y_1, Y_2, ..., Y_N$ be the income or expenditure variable of some population. The Economists are interested in monitoring the welfare of the worse-off in this population. In this extent, poverty measures are defined and used to compare subgroups and to follow the evolution of the poor with respect to time. A poverty measure is assumed to fulfil a number of axiomes since the pionniering work of Sen ([11]). Many authors proposed poverty indices and studied their advantages, like Sen himself, Thon ([15]), Kakwani ([6]), Clark-Hemming-Ulph ([2]), Foster-Greer-Thorbecke ([5]), Ray ([18]), Shorroks ([13]). Most of the required properties for such indices are stated and described in ([17]) along with a broad survey of the available poverty indices.

Asymptotic theories for theses quantities, when they come from random samplings, have been given in recent years. Dia([4]) used point process theory to give asymptotic normality for the Foster-Greer-Thorbecke index. Sall and Lo([10]) studied an asymptotic theory for the poverty intensity defined below and further, Sall, Seck and Lo([8]) proved a larger asymptotic normality for a general measure including the Sen, kakwani, FGT and Shorrocks ones.

Now our aim, here, is to unify the monetary poverty measurements with respect as well to Sen's axiomatic approach as to the asymptotic aspects. We

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point out that poverty may be studied through aspects other than monetary ones as well. It can be viewed through the capabilities to meet basic needs (food, education, health, clothings, lodgings, etc.). In our monetary frame, the main tools are the poverty indices. We give, here, a general poverty index denoted as the General Poverty Indice (GPI), which is aimed to summarize all the known and former ones, in the form

$$GPI=\delta(\frac{A(Q,N,Z)}{NB(Q,N)}\sum_{j=1}^{Q}w(\mu_1 N+\mu_2 Q-\mu_3 j+\mu_4)\ d\left(\frac{Z-Y_{j,N}}{Z}\right)), \tag{1.1}$$

where

$$B(Q,N)=\sum_{j=1}^{Q}w(j),$$

$A(\cdot)$, $B(\cdot)$, $w(\cdot)$, *and* $d(\cdot)$ are given measurable functions, μ_i $(1\leq i\leq 4)$ are given constants, Q is the exact but unknown number of poor persons or poor households in the population $\mathcal{P}$ of size N, Z is the poverty line and finally $Y_{1,N}\leq Y_{2,N}\leq ...\leq Y_{N,N}$ are the ordered incomes or expenditures of the individuals or households of $\mathcal{P}$.

The poverty line Z is defined by economics specialists or governmental authorities so that any individual or household with income (say yearly) less than Z is considered as a poor one. The poverty line determination raises very difficult questions as mentionned and showed in ([7]). We suppose here that Z is known, given and justified by the specialists.

Our approach will permit various research works, as well in the Statistical Mathematics field as in the Economics one. It happens that poverty indices are also somewhat closely connected with economic growth questions. We should find conditions on the functions and the constants in (1.1) so that any kind of needed requirements are met and that the hypotheses imposed by the asymptotic normality are also fulfilled. This may lead to a class of perfect or almost perfect povery measures. In this paper, we concentrate on the description of the GPI and on the asymptotic normality theory. Our best achievement is that (1.1), when sampled, is asymptotically normal for a very broad class of underlying distributions. These results are then specialized for the particular indices. We then begin to describe all the available indices in the frame of (1.1) in the next section. In section 3, we establish the asymptotic normality.

2. How does the GPI include the poverty indices

We begin by making two remarks. First, for almost all the indices, the function $\delta(\cdot)$ is the identity function

$$\forall(u\geq 0),\ \delta(u)=I_d(u)=u.$$

We only noticed one exception in the Clark-Hemming-Uhph (CHUT) index. Secondly, we may divide the poverty indices into non-weighted and weigthed ones. The non weighed measures correspond to those for which the weigth is constant and equal to one :

$$w(\mu_1 N+\mu_2 Q-\mu_3 j+\mu_4)\equiv 1.$$

We begin with them.

2.1. **The non-weighted indices.** First of all, the Foster-Greer-Thorbecke (FGT) index of parameter $\alpha \geq 0$,

$$FGT(\alpha) = \frac{1}{N}\sum_{j=1}^{Q}\left(\frac{Z - Y_{j,N}}{Z}\right)^{\alpha}. \tag{2.1}$$

is obtained from the GPI with

$$\delta = I_d, \quad w \equiv 1, \quad d(u) = u^{\alpha}, \quad and\ A(Q,N,Y) = N.$$

The Ray index defined by (see [18])

$$R(Y,N,Q) = \frac{g}{NZ}\sum_{i=1}^{Q}((Z - Y_{j,N})/g)^{\alpha} \tag{2.2}$$

where

$$g = \frac{1}{Q}\sum_{j=1}^{j=Q}(Z - Y_{j,N}) \tag{2.3}$$

is derived from the GPI with

$$\delta = I_d, \quad w \equiv 1, \quad d(u) = u^{\alpha}, \quad and\ A(Q,N,Y) = Ng^{\alpha-1}/Z.$$

The coefficient $A(Q,N)$ depends here on the income or the expenditure and this is quite an exception among the poverty indices. We may also cite here the Watts index (see [16])

$$P_W = \frac{1}{N}\sum_{j=1}^{j=Q}(\ln Z - lnY_{j,N})$$

with

$$\delta = I_d \quad w \equiv 1 \quad d(u) = u \quad A(Q,N,Y) = N\ln Z\ .$$

But this may be derived from the FGT one as follows. The income Y is transformed into $\ln Y$ and, consequently, the poverty line is taken as $\ln Z$. It follows that

$$W(Y) = FGT(1, \ln Y)$$

for the poverty line $\ln Z$. The case is similar for the Chakravarty index (see [1]), $0 < \alpha < 1$,

$$P_{Ch} = \frac{1}{N}\sum_{j=1}^{j=Q}(1 - (\frac{Y_{j,N}}{Z})^{\alpha}).$$

We may consider it through the FGT class

$$W(Y) = FGT(1, Y^{\alpha})$$

for the poverty line Z^{α}. We are now concerned with weighted indices.

2.2. **The weighted indices.** First, the Kakwani class of poverty measures

$$P_{KAK,N}(k) = \frac{Q}{N\Phi_k(Q)} \sum_{j=1}^{Q} (Q-j+1)^k \left(\frac{Z - Y_{j,N}}{Z}\right), \tag{2.4}$$

where

$$\Phi_k(Q) = \sum_{j=1}^{j=Q} j^k$$

comes from the GPI with

$$\delta = I_d, \quad w(u) \equiv (u), \quad d(u) = u, \quad \mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = 3, \quad \mu_3 = 1 \quad and\ A(N,Q) = Q\ .$$

For $k = 1$, $P_{KAK,N}(1)$ is nothing else but the Sen poverty measure

$$P_{Sen} = \frac{2}{Q(Q+1)} \sum_{j=1}^{Q} (Q-j+1) \left(\frac{Z - Y_{j,N}}{Z}\right). \tag{2.5}$$

As to the Shorrocks index

$$P_{SH,N} = \frac{1}{N^2} \sum_{j=1}^{Q} (2N-2j+1) \left(\frac{Z - Y_{j,N}}{Z}\right), \tag{2.6}$$

it is obtained from the GPI with

$$\delta = I_d, \quad w(u) \equiv (u), \quad d(u) = u, \quad \mu_1 = 2, \quad \mu_2 = 0, \quad \mu_3 = 2, \quad \mu_3 = 1 \quad and\ A(N,Q) = Q$$

and

$$A(N,Q)N = 2Q(N-Q) - Q + Q(Q+1)/2$$

Thon ([15]) proposed the following measure

$$P_{Th} = \frac{2}{N(N+1)} \sum_{j=1}^{Q} (N-j+1) \left(\frac{Z - Y_{j,N}}{Z}\right)$$

which belongs to the GPI family for

$$\delta = I_d, \quad w(u) \equiv u, \quad d(u) = u, \quad \mu_1 = 1, \quad \mu_2 = 0, \quad \mu_3 = 1, \quad \mu_3 = 1$$

and for

$$A(N,Q)/2 = \{Q(N-Q) - Q + Q(Q+1)\}/(N+1).$$

Now, we have the CHU index

$$CHU(\alpha) = \frac{Q}{NZ} \left\{ \frac{1}{Q} \sum_{i=1}^{Q} (Z - Y_{j,N})^{\alpha} \right\}^{1/\alpha}$$

$$\left\{ \frac{1}{N} \frac{Q^{\alpha-1}}{N^{\alpha-1}} \sum_{i=1}^{Q} (\frac{Z - Y_{j,N}}{Z})^{\alpha} \right\}^{1/\alpha} = \delta(J_N)$$

cleary is of the GPI form with

$$\delta = (u) = u^{1/\alpha}, \quad w \equiv 1, \quad d(u) = u^{\alpha} \quad and\ A(Q,N) = N(Q/N)^{\alpha-2}.$$

Not all the poverty indices are derived from the GPI. What preceeds only concerns those based on the poverty gaps

$$(Z - Y_j),\ 1 \leq j \leq Q.$$

We mention one of them in the next subsection.

2.3. **An index not derived from the GPI.** The Takayama ([14]) index

$$P_{Ta} = 1 + \frac{1}{N} - \frac{2}{\mu N^2} \sum_{j=1}^{Q} (n-j+1) Y_{j,N},$$

where μ is the empirical mean of the censored income, cannot be derived from the GPI. The main reason is that, it is not based on the poverty gaps $Z - Y_{j,N}$. It violates the monotonicity axiom which states that the poverty measure increases when one poor individual or household becomes richer.

Now we must study the so-called GPI with respect to the axiomatic approach as well as to the asymptotic theory. We focus in this paper to the general theory of asymptotic normality.

3. Asymptotic normality of the GPI

Let us write the GPI in the form

$$GPI_N = \delta(J_N) \tag{3.1}$$

with

$$J_N = \frac{1}{N} \sum_{j=1}^{Q} c(N,Q,j)\ d\left(\frac{Z - Y_{j,N}}{Z}\right).$$

The GPI is estimated from independent and identically distributed observations of the income or the expenditure $Y_1, Y_2 ....$ with underlying distribution G with lower endpoint $y_0 = \inf\{x, G(x) > 0\} \geq 0$. Given a poverty line Z, we have the (random) number of poor households $q_n = q$ in the sample of size n : $Y_1, Y_2, ..., Y_n$. Its sample value is

$$GPI_n = \delta(J_n)$$

with

$$J_n = \frac{1}{n} \sum_{j=1}^{q} c(n,q,j)\ d\left(\frac{Z - Y_{j,n}}{Z}\right). \tag{3.2}$$

Since Y is an income or expenditure variable, its lower endpoint y_0 is not negative. This allows us to study (3.2) via the transform $X = 1/(Y - y_0)$. Throughout this paper, the distribution function of X is

$$F(\cdot) = G(y_0 + 1/\cdot),$$

whose upper endpoint is then $+\infty$. Hence (3.2) is transformed as

$$J_n = \frac{1}{n} \sum_{j=1}^{q} c(n,q,j)\ d\left(\frac{Z - y_0 - X_{n-j+1,n}^{-1}}{Z}\right). \tag{3.3}$$

We will need conditions on the function d(·) and on the weight c(n,q,j), as in ([8]). First assume that

$(D1)$ $d(\cdot)$ admits a continuous derivative on $]0,1)$.

$(D2)$ $d'(\frac{z-y_0}{z})$ and $d((z-y_0)/z)$ are finite.

For $A(u)=1/F^{-1}(1-u)$, we assume that:

$(C1)$ $A(\cdot)$ is differentiable $(0,1)$ (and its derivatice is denoted $A'(u)=a(u)$.)
$(C2)$ $a(\cdot)$ is continuous on an interval $[a',a'']$ with $0<a'<a''<1$.

$(C3)$ $\exists\ u_0>0, \exists\ \eta>-3/2,\ \forall\ u\in(0,u_0)\,, |a(u)|<C_0\ u^{\eta}\exp(\int_u^1 b(t)t^{-1}dt), with$ $b(t)\to 0\ as\ t\to 0.$

The condition (C3) means that $a(\cdot)$ bounded by a regularly varying function

$$S(u)=C_0\ u^{\eta}\exp(\int_u^1 b(t)t^{-1}dt)$$

of exponent $\eta>-3/2$. As to the function δ, we need it to be differentiable on $]0,+\infty[$, precisely :

(E) There is $\kappa>0$ such that $\delta(\cdot)$ is continuously differentiable on $]0,\kappa]$.

We also need some conditions on the weight $c(\cdot)$. In order to state the hypotheses, we introduce further notation. In fact we use in this paper the representations of the studied random variables $X_i,\ i\geq 1$, by $F^{-1}(1-U_i),\ i\geq 1$, where $U_1,\ U_2,...$ is a sequence of independent random variables uniformly distributed on $(0,1)$. Now let $U_n(\cdot)$ and $V_n(\cdot)$ be the uniform empirical distribution and the empirical quantile function based on $U_i, 1\leq i\leq n$. We have

$$j\geq 1,\quad \frac{j-1}{n}<s\leq\frac{j}{n}\Longrightarrow\frac{j}{n}=U_n(V_n(s)) \tag{3.4}$$

so that

$$j\geq 1,\quad \frac{j-1}{n}<s\leq\frac{j}{n}\Longrightarrow c(n,q,j)=c(n,q,nU_n(V_n(s))\equiv L_n(s). \tag{3.5}$$

Since $U_n(V_n(s))\to s$, as n$\to\infty$, our condition on the weight $c(\cdot)$ is that the function $L_n(\cdot)$ is uniformly bounded by some constant $D>0$ and

$$L_n(s)\to L(s), as\ n\to\infty, \tag{3.6}$$

where $L(\cdot)$ is a non-negative C^1-function on $(0,1)$.

We further require that, $n\to\infty$,

$$\sup_{0\leq s\leq 1}\left|\sqrt{n}(L_n(s)-L(s))-\gamma(s)\sqrt{n}(G_n(Z)-G(Z))\right|=o_p(1) \tag{3.7}$$

for some function $\gamma(\cdot)$. Let us finally put

$$m(s)=L(s)\ d\left(\frac{Z-y_0-1/F^{-1}(1-s)}{Z}\right)ds.$$

We are now able to give our general theorem for the GPI.

Theorem 1. *Suppose that (C1-2-3), (D1-2) and (3.7) hold and let*

$$\mu = \int_0^{G(Z)} \gamma(s) d\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)\ ds.$$

and

$$D = \int_0^{G(Z)} L(s)\ d\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right) ds,$$

Then

$$\sqrt{n}(J_n - D) \to \mathcal{N}(0, \vartheta^2)$$

with

$$\vartheta^2 = \theta^2 + (m(G(Z)) + \mu)^2 G(Z)(1 - G(Z)) + \frac{2(m(G(Z)) + \mu)}{Z} \int_0^{G(Z)} sL(s)h(s)ds$$

and with

$$\theta^2 = Z^{-2} \int_0^{G(Z)} \int_0^{G(Z)} L(u)\ L(v)\ h(u)h(v)(u \wedge v - uv)\ du\ dv$$

where

$$h(s) = a(s)\ d'\left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right).$$

If futhermore $D \in]0, \kappa[$, then

$$\sqrt{n}(GPI_n - \delta(D)) \to \mathcal{N}(N(0, \vartheta^2 \delta'(D)^2)$$

The interest of this paper resides on the particular applications of the theorem for the known indices. Before this, we give the guidelines of the proof.

4. PROOFS OF THE RESULT

All our results will be derived from the lemma below. But, first we place ourselves on a probability space where one version of the so-called Hungarian constructions holds. Namely, Csögö and al. (see [3]) have constructed a probability space holding a sequence of independent uniform random variables U_1, U_2, ... and a sequence of Brownian bridges $B_1, B_2, ...$ such that for each $0 < \nu < 1/2$, *as* $n \to \infty$,

$$\sup_{1/n \leq s \leq 1-1/n} \frac{|\beta_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}) \tag{4.1}$$

and

$$\sup_{1/n \leq s \leq 1-1/n} \frac{|B_n(s) - \alpha_n(s)|}{(s(1-s))^{1/2-\nu}} = O_p(n^{-\nu}), \tag{4.2}$$

where $\{\alpha_n(s) = \sqrt{n}\,(U_n(s) - s)\,, 0 \leqslant s \leqslant 1\}$ is the uniform empirical process and $\{\beta_n(s) = \sqrt{n}\,(s - V_n(s))\,, 0 \leqslant s \leqslant 1\}$ is the uniform quantile process. Thoughout ν will be fixed with $0 < \nu < 1/4$. Now we are able to give the lemma.

Lemma 1. *Suppose that (C1-2-3) and (D1-2) hold and*

$$\sup_{0\leq s\leq 1}\sqrt{n}\,|L_n(s)-L(s)|=O_P(1)\ \text{as}\ n\rightarrow\infty. \tag{4.3}$$

Let

$$D=\int_0^{G(Z)}L(s)\ d\left(\frac{Z-y_0-1/F^{-1}(1-s)}{Z}\right)ds.$$

Then we have the expansion

$$\sqrt{n}(J_n-D)=N_n(1)+N_n(2)+\int_{1/n}^{G(Z)}\sqrt{n}(L_n(s)-L(s))d\left(\frac{Z-y_0-1/F^{-1}(1-s))}{Z}\right)ds+o_P(1)$$

with

$$N_n(1)=\frac{1}{Z}\int_{1/n}^{G(Z)}L(s)B_n(s)h(s)ds \tag{4.4}$$

and

$$N_n(2)=m(G(Z))B_n(G(Z)) \tag{4.5}$$

for

$$m(s)=L(s)\ d\left(\frac{Z-y_0-1/F^{-1}(1-s)}{Z}\right)ds.$$

Proof. This expansion is formulae (4.14) in ([8]). Then, we have the expansion

$$\sqrt{n}(J_n-C_n)=\frac{1}{Z}\int_{1/n}^{G(Z)}L(s)B_n(s)h(s)\ ds+n^{-1/2}L_n(1/n)\ d\left(\frac{Z-y_0-1/F^{-1}(1-U_{1,n})}{Z}\right)$$
$$+\int_{1/n}^{G_n(Z)}\sqrt{n}(L_n(s)-L(s))\ d\left(\frac{Z-y_0-1/F^{-1}(1-V_n(s))}{Z}\right)ds$$
$$+\frac{1}{Z}\int_{G(Z)}^{G_n(Z)}L(s)B_n(s)h(s)ds+\frac{1}{Z}\int_{1/n}^{G_n(Z)}L_n(s)B_n(s)\ (h(\zeta_n(s))-h(s))\ ds$$
$$+\frac{1}{Z}\int_{1/n}^{G_n(Z)}L_n(s)\ (\beta_n\,(s)-B_n(s))\ h(\zeta_n(s))\ ds$$

It is proved in ([8]) that

$$\sqrt{n}(J_n-C_n)=N_n(1)+N_n(2)$$
$$+\int_{1/n}^{G_n(Z)}\sqrt{n}(L_n(s)-L(s))\ d\left(\frac{Z-y_0-1/F^{-1}(1-V_n(s))}{Z}\right)ds+o_P(1).$$

This gives

$$\sqrt{n}(J_n-C_n)=N_n(1)+N_n(2)$$
$$+\int_{1/n}^{G(Z)}\sqrt{n}(L_n(s)-L(s))d\left(\frac{Z-y_0-1/F^{-1}(1-s))}{Z}\right)ds$$
$$+\int_{Gn(Z)}^{G(Z)}\sqrt{n}(L_n(s)-L(s))\ d\left(\frac{Z-y_0-1/F^{-1}(1-V_n(s))}{Z}\right)ds+o_P(1)$$

The condition (4.3) leads to the result. □

We are now able to prove the Theorem.

Proof. Let(Ω,Σ,P) be the probability space on which (4.1) and (4.2) hold. The Lemma together with (4.3), (3.7) and (4.5), imply

$$\sqrt{n}(J_n - D) = N_n(1) + N_n(3) + o_P(1),$$

where $N_n(1)$ is defined in (4.4) and

$$N_n(3) = (m(G(Z)+\mu)\alpha_n(G(Z))+o_P(1) = (m(G(Z)+\mu)B_n(G(Z))+o_P(1). \tag{4.6}$$

The vector $(N_n(1), N_n(3))$ is Gaussian and

$$cov(N_n(1), N_n(3)) = \frac{m(G(Z))+\mu}{Z} E \int_{1/n}^{G(Z)} L(s)h(s)B_n(G(Z))B_n(s)ds \tag{4.7}$$

$$= \frac{m(G(Z))+\mu}{Z} \int_{1/n}^{G(Z)} s\ L(s)\ h(s)\ ds.$$

Then $\sqrt{n}(J_n - D)$ is a linear transform $N_n(1)+N_n(3)$ of the Gaussian vector $(N_n(1), N_n(3))$, plus an $o_P(1)$ term. The variance of this Gaussian term is easily computed through (4.7) and the conclusion follows, that is $\sqrt{n}(J_n - D)$ is asymptotically a centered Gaussian random variable with variance (4.7). □

5. Asymptotic normality of particular indices

5.1. The FGT-like class. This concerns the indices of the form

$$FGT(\alpha) = \frac{1}{N}\sum_{j=1}^{Q} d\left(\frac{Z - Y_{j,N}}{Z}\right).$$

We have here

$$L_n = 1$$

so that

$$\gamma = 0$$

Then

$$\sqrt{n}(J_n - D) \to N(0, \vartheta^2)$$

with

$$\vartheta^2 = \theta^2 + m(G(Z)^2 G(Z)(1-G(Z)) + \frac{2m(G(Z))}{Z}\int_0^{G(Z)} sh(s)ds$$

and

$$D = \int_0^{G(Z)} \left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)^{\alpha} ds.$$

We should remark that the conditions $(D1-D2)$ hold for $d(u) = u^{\alpha}, \alpha \geq 0$.

5.1.1. *The statistics nearby the FGT-class.* This concerns the statistics of the form

$$J_n = \delta(\frac{A(q,n)}{n}\sum_{j=1}^{Q} d\left(\frac{Z-Y_{j,n}}{Z}\right),$$

where we have a random weight not depending on the rank's statistic. We will have two sub-cases.

5.1.2. *The case of CHU's index.* Recall

$$CHU_n(\alpha) = \frac{q}{nZ}\left\{\frac{1}{q}\sum_{i=1}^{q}(Z-Y_{j,n})^{\alpha}\right\}^{1/\alpha}$$

$$= \left\{\frac{1}{n}\frac{q^{\alpha-1}}{n^{\alpha-1}}\sum_{i=1}^{Q}(\frac{Z-Y_{j,n}}{Z})^{\alpha}\right\}^{1/\alpha} = \delta(J_N)$$

We easily get,

$$\sqrt{n}((q/n)^{\alpha-1} - G(Z)^{\alpha-1}) = (\alpha-1)G(Z)^{\alpha-2}\sqrt{n}(G_n(Z) - G(Z)) + o_p(1)$$

$$= (\alpha-1)G(Z)^{\alpha-2}B_n(G(Z)) + o_p(1).$$

By putting

$$C_n = FGT(\alpha) = \frac{1}{n}\sum_{i=1}^{Q}(\frac{Z-Y_{j,n}}{Z})^{\alpha}$$

and

$$C = \int_0^{G(Z)}\left(\frac{Z-y_0-1/F^{-1}(1-s)}{Z}\right)^{\alpha} ds, \tag{5.1}$$

we have, by the general theorem

$$\sqrt{n}(C_n - C) = N_n(1) + N_n(2) + o_p(1)$$

with $L = 1$. By combining these formalae, we get

$$\sqrt{n}(J_n - G(Z)^{\alpha-1}C) \to N(0, \zeta^2)$$

with

$$\zeta^2 = \theta^2 + H(1-G(Z))\int_0^{G(Z)} s\ a(s)ds + H^2G(Z)(1-G(Z))/2$$

where

$$H = C(\alpha-1) + G(Z)m(G(Z))G(Z)^{\alpha-2}.$$

Finally, we get

$$\sqrt{n}(CHU_n(\alpha) - \delta(G(Z)^{\alpha-1}C) \to N(0, (\zeta\delta'(G(Z)^{\alpha-1}C)^2),$$

where

$$\delta'(G(Z)^{\alpha-1}C)^2 = G(Z)^{-(\alpha-1)^2/\alpha}C^{(1-\alpha)/\alpha}.$$

5.1.3. *The case of Ray's index.* Recall

$$R_n = \frac{g}{nZ}\sum_{i=1}^{q}((Z - Y_{j,n})/g)^{\alpha} \tag{5.2}$$

where

$$g = \frac{1}{q}\sum_{j=1}^{j=q}(Z - Y_{j,n}). \tag{5.3}$$

We have

$$J_n = g^{\alpha-1} \times C_n$$

with

$$C_n = FGT_n(\alpha)$$

and

$$C(\alpha) = \int_0^{G(Z)} \left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z}\right)^{\alpha} ds.$$

We use the notation for the CHU index and we also get (5.1). But

$$g = \frac{Zn}{q} \times \frac{1}{n}\sum_{j=1}^{j=q}(\frac{Z - Y_{j,n}}{Z}) = \frac{Zn}{q}K_n.$$

We also have

$$\sqrt{n}(K_n - K) = \frac{1}{Z}\int_{1/n}^{G(Z)} B_n(s)a(s)ds + m_1(G(Z))B_n(G(Z)) + o_p(1)$$

with

$$K = C(1) = \int_0^{G(Z)} \frac{Z - y_0 - 1/F^{-1}(1-s)}{Z} ds.$$

and

$$\sqrt{n}(\frac{Zn}{q} - ZG(Z)^{-1}) = Z\sqrt{N}(G(Z) - G_n(Z))G(Z)^{-2} + o_p(1)$$
$$= -Z\sqrt{N}B_n(G_n(Z))G(Z)^{-2} + o_p(1).$$

By combining all that preceeds, we arrive at

$$\sqrt{n}(g - KZG(Z)^{-1}) = (m_1(G)ZG(Z)^{-1} - KZG(Z)^{-2})B_n(G(Z))$$
$$+\frac{1}{G(Z)}\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1)$$

and

$$\sqrt{n}(g^{\alpha-1} - (KZ/G(Z))^{\alpha-1}) = (\alpha-1)(KZ/G(Z))^{\alpha-2}$$
$$= (\alpha-1)(KZ/G(Z))^{\alpha-2}\ (m_1(G)ZG(Z)^{-1} - KZG(Z)^{-2})B_n(G(Z))$$
$$+\frac{(\alpha-1)(KZ/G(Z))^{\alpha-2}}{G(Z)}\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1)$$
$$= R_1B_n(G(Z)) + R_2\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1).$$

Finally

$$\sqrt{n}(R_n - (KZ/G(Z))^{\alpha-1})C) =$$

$$\frac{(KZ/G(Z))^{\alpha-1})}{Z}\int_{1/n}^{G(Z)} B_n(s)h(s)ds + (KZ/G(Z))^{\alpha-1})\ m_\alpha(G(Z))B_n(G(Z))$$

$$+CR_1B_n(G(Z)) + CR_2\int_{1/n}^{G(Z)} B_n(s)a(s)ds + o_p(1)$$

$$= \int_{1/n}^{G(Z)} B_n(s)\psi(s)ds + \left\{(KZ/G(Z))^{\alpha-1})\ m_\alpha(G(Z)) + CR_1\right\} B_n(G(Z)) + o_P(1)$$

$$= \int_{1/n}^{G(Z)} B_n(s)a(s)ds + A_2B_n(G(Z)) + o_P(1),$$

with

$$\psi(s) = a(s)\left\{C(\alpha)R_2 + (KZ/G(Z))^{\alpha-1}Z^{-1}d'(Z^{-1}(Z-y_0-1/F^{-1}(1-s)))\right\}.$$

Notice that $\int_{1/n}^{G(Z)} B_n(s)h(s)ds + A_1B_n(G(Z))$ is a normal centered random variable with variance

$$\xi^2 = \int_0^{G(Z)}\int_0^{G(Z)} \psi(u)\psi(v)(u\wedge v-uv)\ du\ dv + A_1^2G(Z)(1-G(Z)) + 2A_1(1-G(Z))\int_0^{G(Z)} s\ \psi(s)ds.$$

We conclude that

$$\sqrt{n}(R_n - (KZ/G(Z))^{\alpha-1})C) \to_d N(0,\xi^2)$$

with

$$m_\alpha(u) = (Z^{-1}(Z-y_0-1/F^{-1}(1-s))^\alpha,$$

$$R_1 = (\alpha-1)(KZ/G(Z))^{\alpha-2}\ (m_1(G)ZG(Z)^{-1} - KZG(Z)^{-2}),$$

$$R_2 = (\alpha-1)(KZ/G(Z))^{\alpha-2}G(Z)^{-1},$$

and

$$A_1 = (KZ/G(Z))^{\alpha-1})\ m_\alpha(G(Z)) + CR_1$$

5.2. **The Shorrocks-like indices.** This concerns the Thon and Shorrocks measures. They both have a similar asymptotic behavior.

For Shorrocks's index, we have

$$P_{SH,N} = \frac{1}{n^2}\sum_{j=1}^{n}(2n-2j+1)\left(\frac{Z-Y_{j,n}}{Z}\right).$$

But

$$j \geq 1,\ \frac{j-1}{n} < s \leq \frac{j}{n} \Longrightarrow L_n(s) = c(n,q,j) = (2 - 2*j/n + 1/n) \tag{5.4}$$

$$\to L(s) = 2(1-s),$$

and,

$$\sqrt{n}(L_n(s) - L(s)) = -2*\sqrt{n}(U_n(V_n(s)) - s) + 1/\sqrt{n},$$

By ([12]), p.151,

$$\sqrt{n}\sup_{0\leq s\leq 1}|L_n(s) - L(s)| \leq 3/\sqrt{n}.$$

and then

$$\gamma \equiv 0,\ h(\cdot) = a(\cdot)$$

For the Thon Statistic,

$$P_T = \frac{2}{n(n+1)} \sum_{j=1}^{q} (n-j+1) \left(\frac{Z - Y_{j,n}}{Z} \right),$$

we also have

$$L(s) = 2(1-s),\ \gamma \equiv 0,\ a(\cdot))h(s).$$

In both cases, we get

$$\sqrt{n}(J_n - D) \rightarrow N(0, \vartheta^2)$$

with

$$D = 2 \int_0^{G(Z)} (1-s) \ \left(\frac{Z - y_0 - 1/F^{-1}(1-s)}{Z} \right) ds,$$

$$\vartheta^2 = \theta^2 + m(G(Z)G(Z)(1-G(Z)) + \frac{4m(G(Z))}{Z} \int_0^{G(Z)} s(1-s)a(s)ds$$

and with

$$\theta^2 = 4Z^{-2} \int_0^{G(Z)} \int_0^{G(Z)} (1-u)(1-v)\ a(u)a(v)(u \wedge v - uv)\ du\ dv.$$

5.3. **The Kakwani-class.** The Kakwani class

$$P_{KAK,n} = \frac{q}{n\Phi_k(q)} \sum_{j=1}^{q} (q-j+1)^k \left(\frac{Z - Y_{j,N}}{Z} \right),$$

is introduced with a positive integer. We consider here that k is merely a non-negative real number. It is proved in ([9]) that

$$L(s) = (k+1)(1 - s/G(Z))^k$$

and that

$$\gamma(s) = k(k+1)(1 - s/G(Z))^{k-1}(s/G(Z)^2).$$

We remark that $m(G(Z)) = 0$. Then our resultat is particularized as

$$\sqrt{n}(P_{KAK,n}(k) - D) \rightarrow N(0, \vartheta^2)$$

with

$$\vartheta^2 = \theta^2 + \mu^2 G(Z)(1-G(Z)) + \frac{2\mu}{Z}(1-G(Z)) \int_0^{G(Z)} sL(s)h(s)ds \tag{5.5}$$

and with

$$\theta^2 = Z^{-2} \int_0^{G(Z)} \int_0^{G(Z)} L(u)\ L(v)\ h(u)h(v)(u \wedge v - uv)\ du\ dv.$$

for a fixed real number $k \geq 1$.

We have now finished the poverty indices' review. Some of these results have been simulated and applied in particular issues with the Senegal Data.

6. Conclusion

The GPI includes most of the poverty indices. We have established here their asymptotic normality with immediate applications to poor countries data for finding accurate confidence intervals of the real poverty measurement. In coming papers, a special study will be devoted to the Takayama statistic. The GPI is to be thoroughly visited through the poverty axiomatic approach as well.

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