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GENERALIZED HILL'S ESTIMATOR

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Abstract

We introduce a statistical process depending on a continuous time parameter τ whose any margin can arise as a Generalized Hill's estimator. Under specific conditions, the strong consistency of this estimator is studied. Its asymptotic variance is given with respect to the value of τ. In some situations, the Generalized Hill's estimator (with $\tau = 1/2$) performs better than the Hill's estimator.

1. Introduction

Suppose $(X_n)_{n\geq 1}$ is a stationary sequence whose marginal, one-dimensional distribution is F and that

$$P[X_1 > x] = 1 - F(x) = x^{-\alpha}L(x), \quad x \to \infty, \tag{1.1}$$

where $\alpha > 0$ and L is a slowly varying function satisfying

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1, \quad \forall x > 0.$$

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Our main goal is to estimate the shape parameter α. A popular estimator of $\gamma = 1/\alpha$ is the Hill's estimator [11] obtained as follows: For $1 \leq i \leq n$, write $X_{i,n}$ for the i-th largest value of $X_1, X_2, \ldots, X_n$. Hill's estimator based on the observations $X_1, X_2, \ldots, X_n$ is

$$H_{k,n} = \frac{1}{k}\sum_{i=1}^{k} \log \frac{X_{n-i+1,n}}{X_{n-k,n}}.$$

Another useful expression of $H_{k,n}$ is given by

$$H_{k,n} = \frac{1}{k}\sum_{i=1}^{k} i \log \frac{X_{n-i+1,n}}{X_{n-i,n}}. \tag{1.2}$$

Note that k is the number of upper order statistics used in the estimation. Hill's estimator is known to be consistent and under reasonable conditions asymptotically normal for iid samples. See [10], [13], [14], [12], [8], [6], [17]. Mason [13] proved weak consistency of Hill's estimator for any sequence $k = k(n) \to \infty$, $k(n)/n \to 0$, as $n \to \infty$ and [3] proved strong consistency for any sequence $k = k(n) \to \infty$, with $k(n)/n \to 0$, $\log\log n/k(n) \to 0$, as $n \to \infty$.

However McCulloch [15] observed that Hill's estimator greatly overestimated the tail parameter α for stable data with $1.5 < \alpha < 2$. Drees et al. [4] reported that the traditional Hill plot as a diagnostic tool is most effective only when the underlying distribution is Pareto or very close to Pareto and suggested a simple device: an Alternative Hill plot called *altHill* plot. The Generalized Hill's estimator that we propose performs better according to the choice of the continuous time parameter τ when the underlying distribution is far from Pareto or stable with $1.5 < \alpha < 2$.

We now introduce a statistical process depending on a continuous time parameter whose any margin can arise as a Generalized Hill's estimator. We define for $\tau > 0$

$$H_{k,n}^{\tau} = \sum_{i=1}^{k} \left(\frac{i}{k}\right)^{\tau} \log \frac{X_{n-i+1,n}}{X_{n-i,n}}. \tag{1.3}$$

This class of estimators is in its formulation a special case of the kernel-type estimators proposed in [2] with $K(u) = u^{\tau-1}1_{\{0<u<1\}}$. Under conditions on the kernel $K(\cdot)$, [2] established weak consistency (under H1, H2, H3, H4) and asymptotic normality (under H5, H6, H7). The estimators that we propose have a kernel which does not satisfy H4, H6, H7 when $\tau \leq 1/2$. In this paper, we establish the strong consistency of the proposed estimator when $\tau > 0$ and we give its asymptotic variance. The study of the asymptotic distribution when $\tau \leq 1/2$ is ongoing.

Remark. When $\tau = 1$, then $H_{k,n}^{\tau}$ is the Hill's estimator.

To give a motivation for $H_{k,n}^{\tau}$ being an appropriate estimator under the model (1.1), it suffices to note that

$$\int_x^{\infty} \frac{(1 - F(y))^{\tau}}{y(1 - F(x))^{\tau}} dy = \int_1^{\infty} \left(\frac{1 - F(xt)}{1 - F(x)}\right)^{\tau} \frac{dt}{t} \to \int_1^{\infty} t^{-\tau\alpha-1} dt = \frac{1}{\tau\alpha}$$

by regular variation of $1 - F$ and dominated convergence. If F is replaced by its natural estimator, i.e., by the empirical distribution function F_n based on $X_1, X_2, \ldots, X_n$ and if x equals $X_{n-k,n}$, then we get

$$H_{k,n}^{\tau} = \int_{X_{n-k,n}}^{\infty} \frac{(1 - F_n(y))^{\tau}}{y(1 - F_n(X_{n-k,n}))^{\tau}} dy.$$

Therefore $\tau H_{k,n}^{\tau}$ arise as an appropriate estimator for α^{-1} for all $\tau > 0$. Let us now introduce some notations and conventions before stating our results. Throughout the paper, we assume $X_i > 1$. The sequence of nonnegative variables $Y_1, Y_2, \ldots$ is related to the X_i's by $Y_i = \log X_i$, $i \geq 1$, $G(y) = F(e^y)$.

$U_1, U_2, \ldots, U_n$ will be a sequence of independent and uniform rv's on $(0, 1)$ and $(U_{1,n} \leq U_{2,n} \leq \cdots \leq U_{n,n})$ will denote the corresponding order statistics. Let $U_n(\cdot)$ be the empirical distribution function based on

$U_1, U_2, \ldots, U_n (n = 1, 2, \ldots)$. We then put for $0 < \tau < \infty$, $n \geq 1$,

$$\tilde{x}_n = G^{-1}(1 - U_{k+1,n}), \quad \tilde{z}_n = G^{-1}(1 - U_{1,n}),$$

$$R(x, z, G) = \int_x^z \frac{1 - G(t)}{1 - G(x)}\, dt, \quad R(x, G) = R(x, x_0(G), G),$$

where $x_0(G) = \sup\{x, G(x) < 1\}$ and $G^{-1}(s) = \inf\{x, G(x) \geq s\}$ denote respectively the right endpoint and the left continuous inverse of G. We may without loss of generality and do assume the general representations for the empirical distribution function (df) G_n based on $Y_1, Y_2, \ldots, Y_n$ and for the statistics $Y_{1,n} \leq \cdots \leq Y_{n,n}$ by their uniform counterparts:

$$\{1 - G_n(x), x \in \mathbb{R}, n \geq 1\} = \{U_n(1 - G(x)), x \in \mathbb{R}, n \geq 1\}, \tag{1.4}$$

$$\{Y_{n-i+1,n}, 1 \leq i \leq n, n \geq 1\} = \{G^{-1}(1 - U_{i,n}), 1 \leq i \leq n, n \geq 1\}. \tag{1.5}$$

In what follows, the strong consistency of the Generalized Hill's estimator is studied in Section 2. The asymptotic variance of the estimator is given in Section 3. In Section 4 we present some simulations for illustrating our main result. The influence of the continuous parameter is also studied in finite samples. Section 5 concludes the paper by providing a summary of the more important findings. The proofs are postponed in an Appendix in Section 6.

2. Consistency

In this section we prove strong consistency of the Generalized Hill's estimator.

Let us begin with some recalls and technical lemmas.

Theorem 1 (Fisher-Tippet theorem, limit laws for maxima)**.** *Let (X_n) be a sequence of iid random variables. If there exist two sequences $(a_n > 0), (b_n)$ and some non-degenerate distribution function H such that*

$$\mathbb{P}\{a_n^{-1}(\max_{1 \leq k \leq n} X_k - b_n) \leq x\} \xrightarrow{d} H(x), \tag{2.1}$$

then H belongs to the type of one of the following standard extreme value distributions:

$$\textit{Gumbel} \quad \Lambda(x) = \exp(-e^{-x}), \quad x \in \mathbb{R},$$

$$\textit{Fréchet} \quad \phi_\alpha(x) = \exp(-x^{-\alpha}), \quad x > 0,\ \alpha > 0,$$

$$\textit{Weibull} \quad \psi_\alpha(x) = \exp(-(-x)^{\alpha}), \quad x \le 0,\ \alpha > 0.$$

A *non degenerate distribution function* H_1 is said to be of the type of H_2 if $H_1(x) = H_2(cx + d)$ for some constants $c > 0$ and $d \in \mathbb{R}$. We say that the df F belongs to the maximum domain of attraction of the extreme value distribution H, if there exist constants $a_n > 0$ and b_n such that (2.1) holds. We write $F \in D(H)$.

Before going further, we give here a useful technical lemma which follows from [13].

Lemma 1. (i) $F \in D(\phi_\alpha), \alpha > 0,$ *if and only if* $G \in D(\Lambda)$ *and* $R(x, G) \to \frac{1}{\alpha}$ *as* $x \to x_0(G) = \infty$.

(ii) *Let H be any distribution function such that* $R(x, H)$ *is finite for* $x < x_0(H)$ *and* $\frac{z - x}{R(x, H)} \to +\infty$ *as* $x \to x_0(H), z \to x_0(H), x < z$.

Then

$$\frac{R(x, z, H)}{R(x, H)} \to 1 \ \textit{as}\ x \to x_0(H),\ z \to x_0(H),\ x < z. \tag{2.2}$$

(iii) *Let* $G_r(x) = 1 - (1 - G(x))^r$, $r > 0$.

If $G \in D(\Lambda)$, *then* $G_r \in D(\Lambda)$ *with* $x_0(G) = x_0(G_r) = y_0$ *and*

$$\frac{R(x, z, G_r)}{R(x, z, G)} \to \frac{1}{r} \ \textit{as}\ x \to y_0,\ z \to y_0. \tag{2.3}$$

The proof of the main result requires the following lemma.

Lemma 2. *If* $F \in D(\phi_\alpha)$, *then* $(1 - G(G^{-1}(1 - u)))/u \to 1$ *as* $u \to 0$.

We now state and prove our main result.

Theorem 2. *Let* $X_1, X_2, \ldots$ *be a sequence of independent and identically distributed* (*iid*) *random variables* (*rv*) *with distribution function F satisfying* (1.1). *Assume that* $X_i > 1$. *Let* $k(n) \to +\infty$, $\frac{k(n)}{n} \to 0$ *and* $\frac{\log\log n}{k(n)} \to 0$ *as* $n \to +\infty$. *If* $\tau > 0$, *then*

$$\tau H_{k,n}^{\tau} \to \frac{1}{\alpha}, \text{ a.s. as } n \to +\infty. \tag{2.4}$$

3. Asymptotic Variance

In this section we discuss the asymptotic variance of the Generalized Hill's estimator with respect to the value of the parameter τ. It is well known that $1 - F$ is regularly varying of order $-\alpha$ in the upper tail, if and only if $F^{-1}(1-s)$ is regularly varying of order $-1/\alpha$ at 0. To give the asymptotic variance of the proposed estimator, we split $H_{k_n,n}^{\tau}$ into three parts:

$$H_{k_n,n}^{\tau} =: A_{1,n} + A_{2,n} + A_{3,n},$$

where

$$A_{1,n} = \frac{1}{\alpha}\sum_{i=1}^{k_n}\left(\frac{i}{k_n}\right)^{\tau}\log\frac{U_{i,n}}{U_{i+1,n}},$$

$$A_{2,n} = \sum_{i=1}^{k_n}\left(\frac{i}{k_n}\right)^{\tau}\log\frac{1+f(U_{i,n})}{1+f(U_{i+1,n})}$$

and

$$A_{3,n} = \sum_{i=1}^{k_n}\left(\frac{i}{k_n}\right)^{\tau}\int_{U_{i,n}}^{U_{i+1,n}}\frac{b(u)}{u}\,du.$$

We shall prove the following result.

Theorem 3. *Let* $X_1, X_2, \ldots$ *be a sequence of independent and identically distributed* (*iid*) *random variables* (*rv*) *with distribution function F satisfying* (1.1). *Assume that* $X_i > 1$. *Then for every sequence* k_n *of positive integers satisfying* $1 \leq k_n \leq n$, $k_n \to +\infty$ *and* $k_n/n \to 0$ *and for every* $\tau > 0$, *we have*

$$H^{\tau}_{k_n, n} = A_{1,n} + o(1) \ a.s.$$

with

$$var(A_{1,n}) = \begin{cases} \mathcal{O}((\log k_n)/k_n), & \text{if } \tau = 1/2, \\ \mathcal{O}(1/k_n^{2\tau}), & \text{if } \tau < 1/2, \\ \mathcal{O}(1/k_n), & \text{if } \tau > 1/2. \end{cases} \tag{3.1}$$

4. Simulation Study

Now we investigate, by simulation experiments, the convergence of the Generalized Hill's estimator given in Theorem 2 for different values of the parameter τ. In the sequel, we will choose $\tau = 1$ (corresponding to the classical Hill's estimator), $\tau = 0.5$ and $\tau = 5$. The three estimators are called respectively *Hill*(1), *Hill*(0.5) and *Hill*(5).

4.1. Generalized Hill plot

A Hill plot is a plot of $\{(k, H^{-1}_{k,n}), 1 \leq k \leq n-1\}$, where $H_{k,n}$ is the Hill's estimator defined in (1.2) which is construct from the k largest order statistics $X_{i,n}$, $1 \leq i \leq n$ of a sample size n (see [4]). The estimate of α is $H^{-1}_{k,n}$. Similarly we defined the Generalized Hill plot as

$$\{(k, (\tau H^{\tau}_{k,n})^{-1}), 1 \leq k \leq n-1\}$$

and infer the value of α from a stable region in the graph. Figures 1-4 give Generalized Hill's estimator plots as a function of the number of order statistics. In each graph, the dotted line represents the true value of

α. The left graph applies the Hill's estimator $(\tau = 1)$ to 5000 iid observations. The middle graph applies the Generalized Hill's estimator $(\tau = 0.5)$ to the same set of data. The right graph gives the Generalized Hill plot with $\tau = 5$ for the same set of data. Figure 1 is a Generalized Hill plot for iid observations from the Pareto distribution with $\alpha = 1$. Figure 2 shows a Generalized Hill plot for iid observations from distribution F satisfying $1 - F(x) \sim x^{-1}/\log x$. Figures 3 and 4 display samples of size 5000 from stable distributions with parameters $\alpha = 0.5$ and $\alpha = 1.75$.

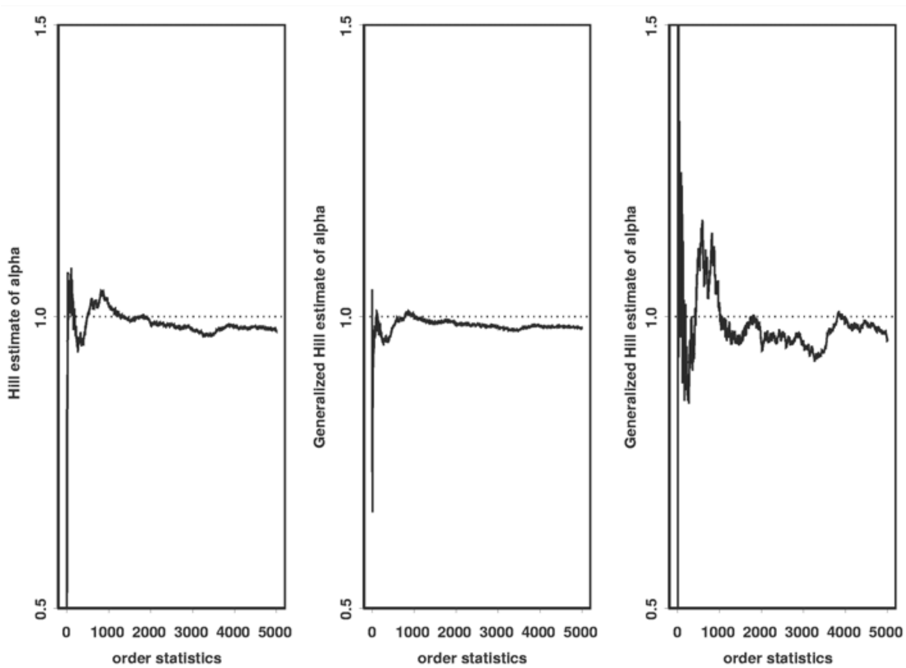

Figure 1. Generalized Hill plot of 5000 Pareto observations with $\alpha = 1$. Classical Hill plot (left), Generalized Hill plot with $\tau = 0.5$ (middle) and $\tau = 5$ (right).

The Generalized Hill plot can be seen as a simple device like the *altHill* plot as suggested by [18].

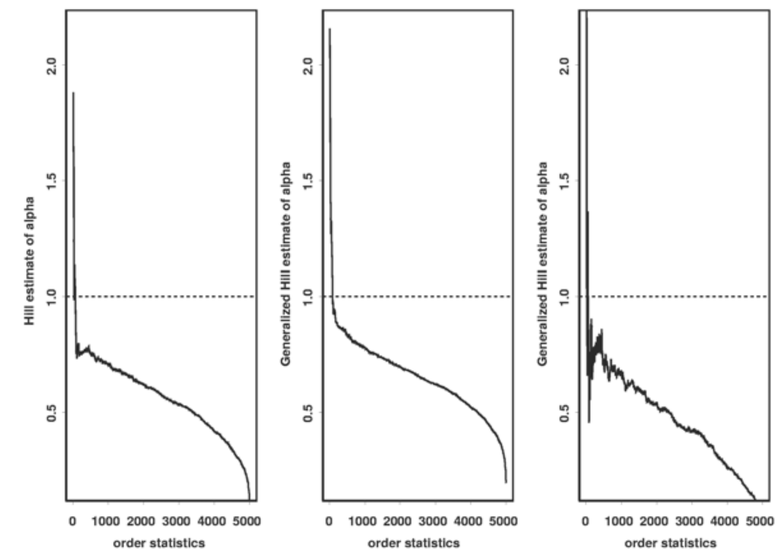

Figure 2. Generalized Hill plot of 5000 observations from distribution F satisfying $1 - F(x) \sim x^{-1}/\log x.$ Classical Hill plot (left), Generalized Hill plot with $\tau = 0.5$ (middle) and $\tau = 5$ (right).

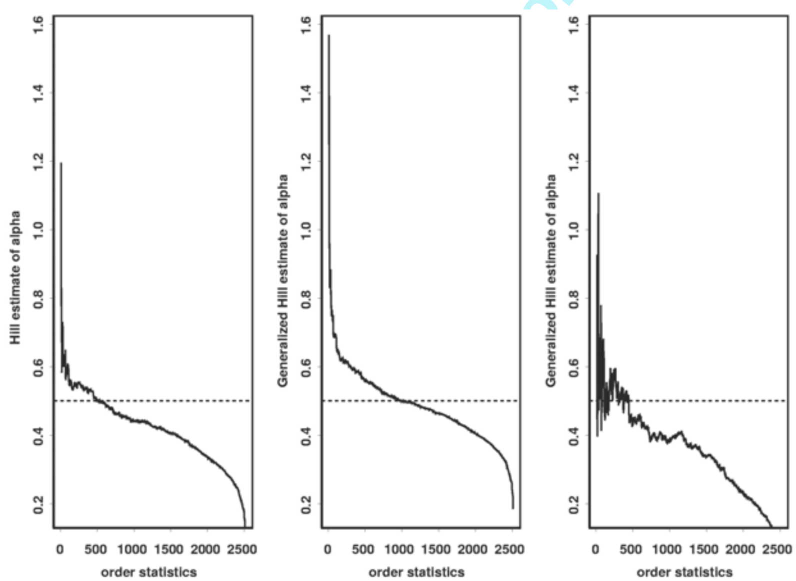

Figure 3. Generalized Hill plot of 5000 observations from stable distribution with $\alpha = 0.5.$ Classical Hill plot (left), Generalized Hill plot with $\tau = 0.5$ (middle) and $\tau = 5$ (right).

4.2. The influence of the parameter τ

Tables 1-4 give the summary statistics for the estimator $(\tau H_{k,n}^{\tau})^{-1}$ based on $m = 1000$ data sets of size $n = 20000$ from four different distributions when the number k of order statistics used in the estimation is set to 250 and 500 respectively.

We plotted on Figures 5-8 the boxplot of the observed sampling distribution of $(\tau H_{k,n}^{\tau})^{-1}$ from the four underlying distributions. In order to test the influence of the value of the parameter τ, we have used $\tau = 1, 0.5, 5$ corresponding respectively to *Hill*(1), *Hill*(0.5), *Hill*(5).

We summarize the findings as follows. As expected, Hill's estimator performs better when the underlying distribution is Pareto or very close to Pareto or stable with index $0 < \alpha \leq 1.5$. See Table 1, Table 3, Figure 5 and Figure 7. There is an obvious influence of the tail thickness of the underlying distribution in the comparisons of the estimators. In Table 2, the value obtained for the mean, for instance, is lower than 1. This means that the estimator $(\tau H_{k,n}^{\tau})^{-1}$ underestimate the index α when the underlying distribution is heavy tailed. However Hill(0.5) seems to perform better than Hill(1). Similarly to the Shifted Hill's estimator introduced by [1], the Generalized Hill's estimator with $\tau = 0.5$ is more satisfactory to estimate the tail thickness parameter when the underlying distribution is stable with $1.5 < \alpha < 2$, (see Table 2 and Figure 6) although the asymptotic variance of Hill(0.5) is a bit greater than the one of Hill(1).

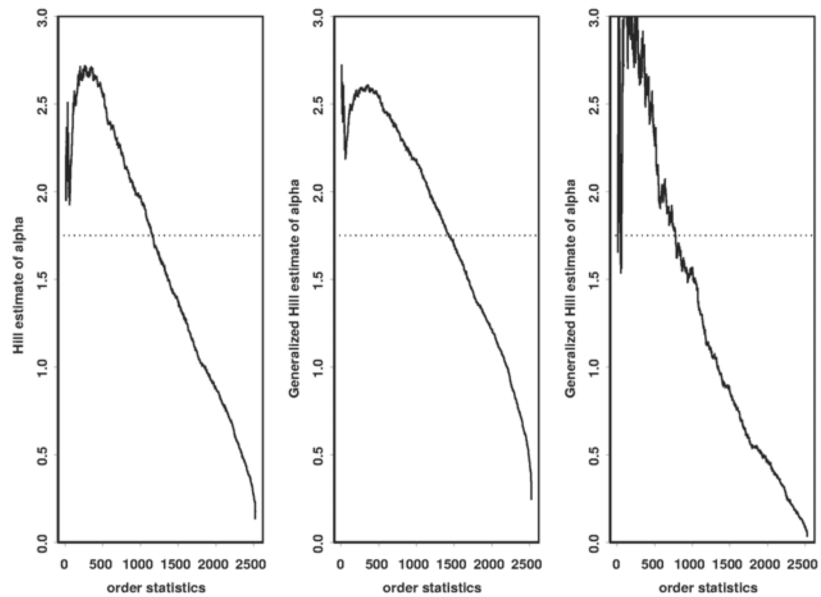

Figure 4. Generalized Hill plot of 5000 observations from stable distribution with $\alpha = 1.75$. Classical Hill plot (left), Generalized Hill plot with $\tau = 0.5$ (middle) and $\tau = 5$ (right).

This is the price of robustness.

Table 1. Summary Statistics for $(\tau H_{k,n}^{\tau})^{-1}$ based on $m = 1000$ data sets of size $n = 20000$ from a Pareto distribution with $\alpha = 1$

	$k = 250$			$k = 500$		
	$\tau = 1$	$\tau = 0.5$	$\tau = 5$	$\tau = 1$	$\tau = 0.5$	$\tau = 5$
Statistic	Hill(1)	Hill(0.5)	Hill(5)	Hill(1)	Hill(0.5)	Hill(5)
Minimum	0.842	0.747	0.677	0.842	0.794	0.761
Lower quartile	0.955	0.991	0.919	0.964	0.991	0.940
Median	0.998	1.049	0.986	0.994	1.030	0.987
Upper quartile	1.038	0.108	0.0013	1.024	1.072	1.039
Maximum	1.235	1.361	1.432	1.167	1.241	1.267
Mean	0.999	1.050	0.992	0.995	1.031	0.990
Std dev	0.063	0.085	0.106	0.045	0.061	0.074

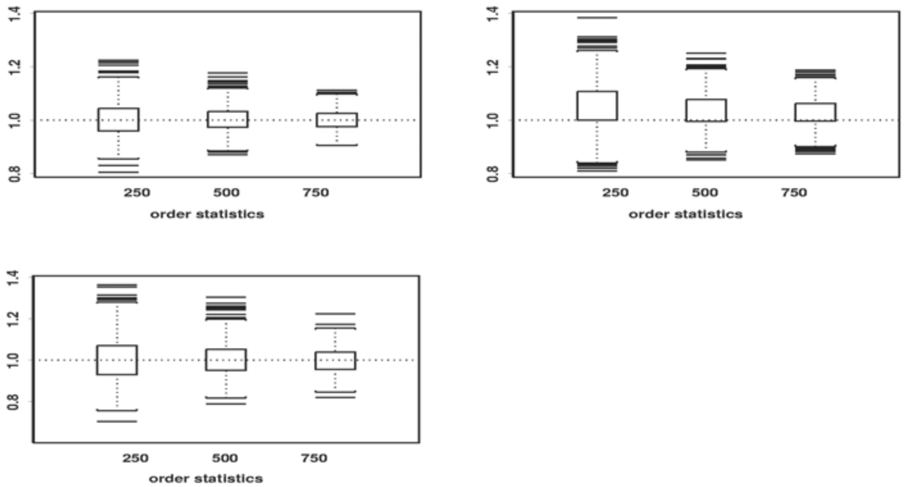

Figure 5. Observed sampling distribution of $(\tau H^{\tau}_{k,n})^{-1}$ based on $m = 1000$ simulated data sets of size $n = 20000$ from a Pareto distribution with $\alpha = 1$. Box plots for $\tau = 1$ (left, top); $\tau = 0.5$ (right, top); $\tau = 5$ (bottom).

Table 2. Summary Statistics for $(\tau H^{\tau}_{k,n})^{-1}$ based on $m = 1000$ data sets of size $n = 20000$ from a distribution F satisfying $1 - F(x) \sim x^{-1}/\log x$

	$k = 250$			$k = 500$		
	$\tau = 1$	$\tau = 0.5$	$\tau = 5$	$\tau = 1$	$\tau = 0.5$	$\tau = 5$
Statistic	Hill(1)	Hill(0.5)	Hill(5)	Hill(1)	Hill(0.5)	Hill(5)
Minimum	0.697	0.675	0.590	0.714	0.699	0.636
Lower quartile	0.810	0.847	0.765	0.797	0.831	0.756
Median	0.844	0.897	0.820	0.820	0.865	0.792
Upper quartile	0.876	0.945	0.877	0.846	0.902	0.834
Maximum	1.053	1.150	1.195	0.966	1.035	0.987
Mean	0.844	0.897	0.822	0.822	0.866	0.796
Std dev	0.052	0.071	0.085	0.037	0.051	0.058

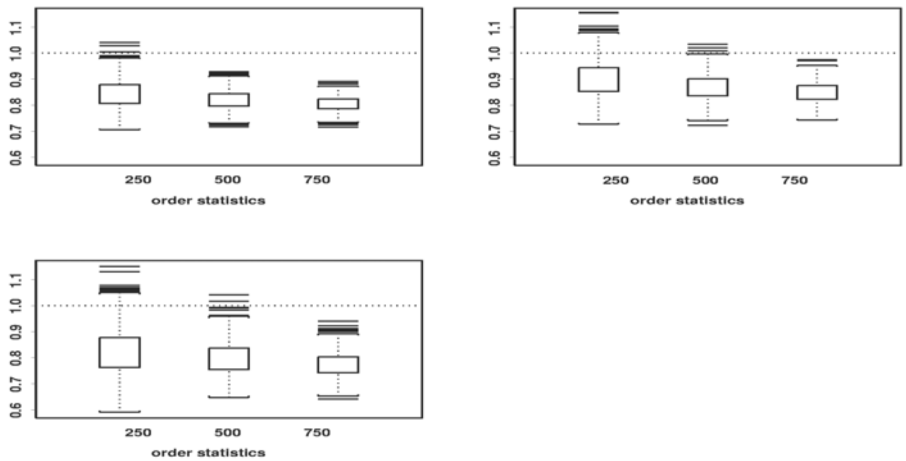

Figure 6. Observed sampling distribution of $(\tau H_{k,n}^{\tau})^{-1}$ based on $m = 1000$ simulated data sets of size $n = 20000$ from a distribution F satisfying $1 - F(x) \sim x^{-1}/\log x$. Box plots for $\tau = 1$ (left, top); $\tau = 0.5$ (right, top); $\tau = 5$ (bottom).

Table 3. Summary Statistics for $(\tau H_{k,n}^{\tau})^{-1}$ based on $m = 1000$ data sets of size $n = 20000$ from a stable distribution F with $\alpha = 0.5$

	$k = 250$			$k = 500$		
	$\tau = 1$	$\tau = 0.5$	$\tau = 5$	$\tau = 1$	$\tau = 0.5$	$\tau = 5$
Statistic	Hill(1)	Hill(0.5)	Hill(5)	Hill(1)	Hill(0.5)	Hill(5)
Minimum	0.417	0.403	0.362	0.432	0.421	0.393
Lower quartile	0.476	0.497	0.457	0.479	0.493	0.463
Median	0.496	0.525	0.491	0.494	0.514	0.488
Upper quartile	0.519	0.551	0.527	0.509	0.535	0.514
Maximum	0.590	0.638	0.702	0.575	0.604	0.648
Mean	0.498	0.525	0.494	0.494	0.514	0.490
Std dev	0.031	0.041	0.055	0.021	0.030	0.037

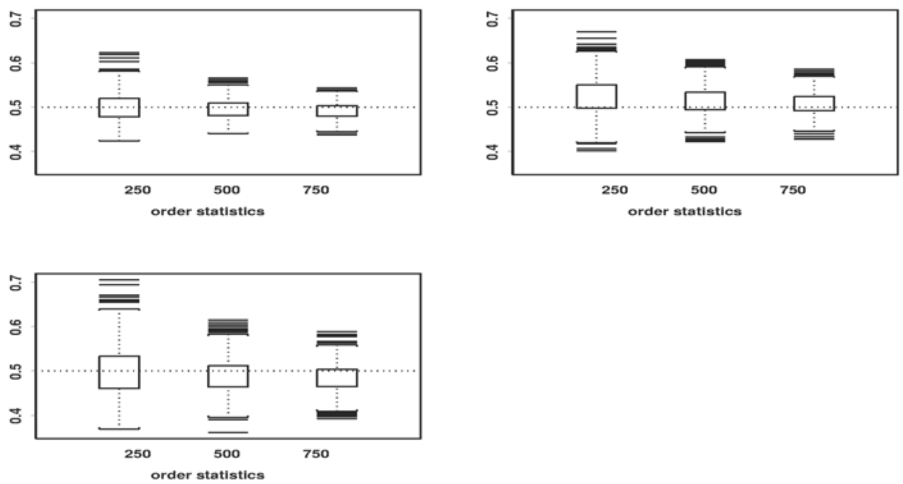

Figure 7. Observed sampling distribution of $(\tau H_{k,n}^{\tau})^{-1}$ based on $m = 1000$ simulated data sets of size $n = 20000$ from a stable distribution F with $\alpha = 0.5$. Box plots for $\tau = 1$ (left, top); $\tau = 0.5$ (right, top); $\tau = 5$ (bottom).

Table 4. Summary Statistics for $(\tau H_{k,n}^{\tau})^{-1}$ based on $m = 1000$ data sets of size $n = 20000$ from a stable distribution F with $\alpha = 1.75$

	$k = 250$			$k = 500$		
	$\tau = 1$	$\tau = 0.5$	$\tau = 5$	$\tau = 1$	$\tau = 0.5$	$\tau = 5$
Statistic	Hill(1)	Hill(0.5)	Hill(5)	Hill(1)	Hill(0.5)	Hill(5)
Minimum	1.659	1.504	1.722	2.041	1.740	2.215
Lower quartile	2.013	1.970	2.220	2.294	2.136	2.600
Median	2.111	2.087	2.398	2.374	2.237	2.746
Upper quartile	2.211	2.220	2.562	2.456	2.348	2.893
Maximum	2.629	2.700	3.221	2.824	2.750	3.424
Mean	2.115	2.098	2.396	2.377	2.243	2.752
Std dev	0.150	0.190	0.256	0.123	0.161	0.206

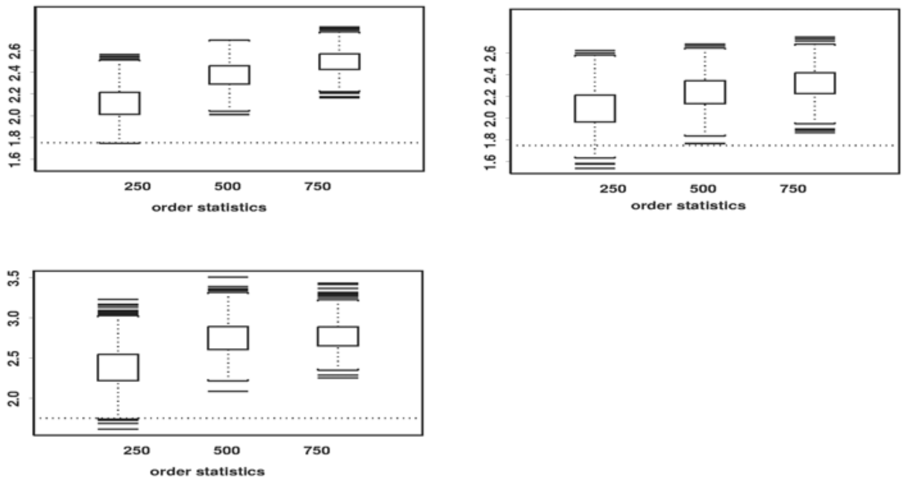

Figure 8. Observed sampling distribution of $(\tau H_{k,n}^{\tau})^{-1}$ based on $m = 1000$ simulated data sets of size $n = 20000$ from a stable distribution F with $\alpha = 1.75$. Box plots for $\tau = 1$ (left, top); $\tau = 0.5$ (right, top); $\tau = 5$ (bottom).

5. Conclusion

In this paper we introduce a statistical process depending on a continuous time parameter τ whose any margin can arise as a Generalized Hill's estimator. Its strong consistency is studied. Our findings indicate that the Generalized Hill's estimator with $\tau = 0.5$ performs better when the underlying distribution is far from Pareto or stable with index $1.5 < \alpha < 2$.

6. Appendix

Proof of Lemma 1. (a) Part (i) may be easily derived from results of Mason [13].

(b) Direct computations give

$$R(x, z, H) = R(x, H)(1 - R_1) \text{ and } 0 \le R_1 \le \left(1 + \frac{z - x}{R(z, H)}\right)^{-1}. \qquad (6.1)$$

The proof of part (ii) is now immediate.

(c) The proof of part (iii) follows directly from (ii) and Theorem 2.8.1 of [7].

Proof of Lemma 2. It is easy checked that either $G(G^{-1}(1-u)) = 1-u$ or $1-u$ lies on some constancy interval of G^{-1}, say $]G(x-), G(x)]$, so that $G^{-1}(1-u) = x$ and hence $1 \geq (1-G(G^{-1}(1-u)))/u \geq 1-G(x)/1-G(x-)$. Since $G \in D(\Lambda)$, by de Haan-Balkema's representation (see [19]),

$$1 - G(x) = c(x)\exp\left(\int_{-\infty}^{x} l(t)^{-1}dt\right), \quad -\infty < x < y_0,$$

where $c(x) \to 1$ as $x \to y_0$ and l admits a derivative $l'(x)$ such that $l'(x) \to 0$ as $x \to y_0$. Hence

$$1 - G(x)/1 - G(x-) = c(x)/c(x-) \to 1 \text{ as } x \to y_0.$$

This completes the proof.

Proof of Theorem 2.

Step 1. Using the representation (1.5), we can write

$$H_{k,n}^{\tau} = \left(\frac{n}{k}\right)^{\tau} \sum_{i=1}^{k} \left(\frac{i}{n}\right)^{\tau} [G^{-1}(1-U_{i,n}) - G^{-1}(1-U_{i+1,n})]$$

which by (1.4) is equal to

$$\left(\frac{n}{k}\right)^{\tau} \int_{\tilde{x}_n}^{\tilde{z}_n} U_n^{\tau}(1-G(t))dt.$$

Let us split $H_{k,n}^{\tau}$ into two parts

$$H_{k,n}^{\tau} = \left(\frac{n}{k}\right)^{\tau} \int_{\tilde{x}_n}^{\tilde{z}_n} (1-G(t))^{\tau} dt + \left(\frac{n}{k}\right)^{\tau} \int_{\tilde{x}_n}^{\tilde{z}_n} U_n^{\tau}(1-G(t)) - (1-G(t))^{\tau} dt.$$

Theorem 4 in [20] implies $U_{k+1,n} \sim \frac{k}{n}$ a.s. as $n \to +\infty$. Observing that $1 - G(\tilde{x}_n) = 1 - G(G^{-1}(1-U_{k+1,n})) \sim U_{k+1,n}$ as $n \to +\infty$, we have

$$1 - G(\tilde{x}_n) = \frac{k}{n}(1 + o(1)), \text{ a.s.}$$

Therefore

$$H_{k,n}^{\tau} = R(\widetilde{x}_n, \widetilde{z}_n, G_\tau)(1 + o(1)) + \left(\frac{n}{k}\right)^{\tau} \int_{\widetilde{x}_n}^{\widetilde{z}_n} U_n^{\tau}(1 - G(t)) - (1 - G(t))^{\tau} dt. \quad (6.2)$$

Step 2. Now we show that

$$R(\widetilde{x}_n, \widetilde{z}_n, G_\tau) \to \frac{1}{\tau\alpha} \text{ as } n \to +\infty. \quad (6.3)$$

Since $F \in D(\phi_\alpha)$, by the Karamata's representation, we have

$$F^{-1}(1 - u) = c(1 + f(u))u^{-1/\alpha} \exp\left(\int_u^1 t^{-1}b(t)dt\right), \quad 0 < u < 1,$$

where $\sup(|f(u)|, |b(u)|) \to 0$ as $u \to 0$ and c is a finite positive constant.

Hence

$$G^{-1}(1 - u) = -\frac{1}{\alpha} \log u + \log c + \log(1 + f(u)) + \int_u^1 t^{-1}b(t)dt$$

which implies that

$$\widetilde{z}_n - \widetilde{x}_n = -\frac{1}{\alpha} \log \frac{U_{1,n}}{U_{k_n+1,n}} + \log \frac{1 + f(U_{1,n})}{1 + f(U_{k_n+1,n})} + \int_{U_{1,n}}^{U_{k_n+1,n}} t^{-1}b(t)dt.$$

Let $0 < \varepsilon < 1/\alpha$. Then there exists an n_0 such that for all $n > n_0$ and for all $t \in (U_{1,n}, U_{k_n+1,n})$, we have

$$\widetilde{z}_n - \widetilde{x}_n \geq \left(\frac{1}{\alpha} - \varepsilon\right) \log \frac{U_{k_n+1,n}}{U_{1,n}} + \log \frac{1 + f(U_{1,n})}{1 + f(U_{k_n+1,n})}.$$

Using the fact that $U_{k_n+1,n} \sim k/n$ a.s. we get $\widetilde{z}_n - \widetilde{x}_n \to \infty$, as $n \to \infty$. Now, by Lemma 1 and Lemma 2, (6.3) follows immediately.

Step 3. It remains to show that the second term of (6.2) which we denote by R_n tends a.s. to 0 as $n \to \infty$. First we note that if $\tau > 1$, then we have

$$||a|^{\tau} - |b|^{\tau}| \leq \tau 2^{\tau-1}|a - b|^{\tau} + \tau 2^{\tau-1}|b|^{\tau-1}|a - b|. \quad (6.4)$$

We write

$$|R_n| \le \tau 2^{\tau-1}\left(\frac{n}{k_n}\right)^{\tau}\int_{\tilde{x}_n}^{\tilde{z}_n}|U_n(1-G(t))-(1-G(t))|^{\tau}dt$$

$$+\ \tau 2^{\tau-1}\left(\frac{n}{k_n}\right)^{\tau}\int_{\tilde{x}_n}^{\tilde{z}_n}|1-G(t)|^{\tau-1}|U_n(1-G(t))-(1-G(t))|dt$$

$$\le: A + B.$$

By Theorem 1(ii) in [5], the following factor is bounded a.s. for $0 < \nu < 1/2$,

$$\left(\frac{n}{k_n}\right)^{\nu}\left(\frac{n}{\log\log n}\right)^{1/2}\sup_{0<t\le\frac{k_n}{n}}\frac{|U_n(t)-t|}{t^{1/2-\nu}} < C_1. \tag{6.5}$$

The equation (6.5) remains valid when we replace k by $[nU_{k_n+1,n}]+1$, where $[\cdot]$ stands for the integer part. Therefore

$$A \le \tau 2^{\tau-1}\left(\frac{n}{k_n}\right)^{\tau}\int_{\tilde{x}_n}^{\tilde{z}_n}\frac{|U_n(1-G(t))-(1-G(t))|^{\tau}}{(1-G(t))^{(1/2-\nu)\tau}}(1-G(t))^{(1/2-\nu)\tau}dt$$

$$\le C_1\tau 2^{\tau-1}\left(\frac{n}{k_n}\right)^{\tau}\left(\frac{[nU_{k_n,n}]+1}{n}\right)^{\tau\nu}\left(\frac{\log\log n}{n}\right)^{\frac{\tau}{2}}\int_{\tilde{x}_n}^{\tilde{z}_n}(1-G(t))^{(1/2-\nu)\tau}dt$$

$$\le C_1\tau 2^{\tau-1}\left(\frac{[nU_{k_n,n}]+1}{k_n}\right)^{\tau\nu}\left(\frac{\log\log n}{k_n}\right)^{\tau/2}R(\tilde{x}_n, \tilde{z}_n, G_{(1/2-\nu)\tau})(1+o(1)). \tag{6.6}$$

Using again Theorem 1(ii) in [5], we prove that

$$\left(\frac{n}{[nU_{k_n,n}]+1}\right)^{1/2}\left(\frac{n}{\log\log n}\right)^{1/2}\sup_{0<t\le U_{k_n+1,n}}|U_n(t)-t| < C_2 \tag{6.7}$$

and we write

$$B \le C_2\tau 2^{\tau-1}\left(\frac{[nU_{k_n+1,n}]+1}{k_n}\right)^{1/2}\left(\frac{\log\log n}{k_n}\right)^{1/2}R(\tilde{x}_n, \tilde{z}_n, G_{\tau-1})(1+o(1)). \tag{6.8}$$

The inequalities (6.6) and (6.8) demonstrate that both A and B converge almost surely to zero when $\log\log n/k_n \to 0$. On the other hand, if $0 < \tau \leq 1$, then we have for $a, b \in \mathbb{R}$

$$\| a |^{\tau} - | b |^{\tau} | \leq | a - b |^{\tau}.$$

This situation is handled very similarly as the first one.

Proof of Theorem 3. First, using the fact that when the random variable $U \sim \mathcal{U}(0, 1)$, then $V = -\log U$ is a standard exponential rv and from Theorem 1.6.1 of [16, p. 37], we have

$$\left\{-\log \frac{U_{i,n}}{U_{i+1,n}}, 1 \leq i \leq n\right\} \stackrel{d}{=} \left\{\frac{E_i}{i}, 1 \leq i \leq n\right\},$$

where $(E_i)_{1 \leq i \leq n}$ are standard rvs. Hence

$$A_{1,n} \stackrel{d}{=} \frac{1}{\alpha k_n} \sum_{i=1}^{k_n} \left(\frac{i}{k_n}\right)^{\tau-1} E_i.$$

Next, notice that $\frac{k_n}{n} \to 0$ implies $U_{k_n+1,n} \to 0$ a.s. Thus since $f(s) \to 0$ as $s \to 0$, we have $A_{2,n} \to 0$ a.s. whenever $\frac{k_n}{n} \to 0$. Finally, since $b(s) \to 0$ as $s \to 0$, we get for large enough n,

$$\left| \int_{U_{i,n}}^{U_{i+1,n}} \frac{b(u)}{u} du \right| \leq \varepsilon \log \frac{U_{i+1,n}}{U_{i,n}}.$$

We have

$$| A_{3,n} | \leq \frac{\varepsilon}{k_n^{\tau}} \sum_{i=1}^{k_n} i^{\tau} \log \frac{U_{i+1,n}}{U_{i,n}} = \frac{\varepsilon}{k_n^{\tau}} \sum_{i=1}^{k_n} (i^{\tau} - (i-1)^{\tau}) \log \frac{U_{k_n+1,n}}{U_{i,n}}.$$

An easy argument similar to those of [3] based on the fact that $\limsup_{n \to +\infty} A_{1,n} < +\infty$ shows that $A_{3,n} \to 0$ as $n \to +\infty$. Now a straightforward calculation of $var(A_{1,n})$ gives the desired result.

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