ON A CONTINUOUS HILL STATISTIC PROCESS AND ITS ASYMPTOTIC NORMALITY.

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Abstract. Let $X_1, X_2, ...$ be a sequence of independent random variables identically distributed with $\mathbb{P}(X_1 \leq x) = F(x)$ and $F(1) = 0$ and let $X_{1,n} \leq ...... \leq X_{n,n}$ the order statistics related to the n$\geq$ 1 first of them. We introduce the following statistics family guided by the continuous parameter τ, $0 < \tau < \infty$,

$$T_n(\tau) = k^{-\tau} \sum_{j=\ell+1}^{j=k} i^{\tau} \left(\log\ X_{n-j+1,n} - \log X_{n-j,n}\right)$$

[where l, k and n are integers such that $0 < l < k < n \geq 1, \quad l/k \to 0$ and $k/n \to 0$ as $n \to \infty$]. This class consists of a continuous generalization of the Hill statistic which is $T_n(1)$. In this paper, the asymptotic normality of the finite-distributions of this familly are completey characterized for $\tau > 1/2$. Related statistical simulations and efficient tests for extremal domains are then proposed.

Key words and phrases : Order Statistics, empirical and quantile processes, Brownian bridges, extremal domain of attraction, asymptotic normality, normal approximations.

1. INTRODUCTION

Let $X_1, X_2, ...$ be a sequence of independent and indentically distributed with $\mathbb{P}(X_1 \leq x) = F(x)$ and $F(1) = 0$. The estimation of the law of the maximum $X_{n,n} = \max(X_1, ..., X_n)$ is one of the most important questions of the extreme value theory. Recall that $X_{n,n}$ converges in type to some random variable Z with non degenerated distribution function $H(.)$ if and only if there exists two sequences of real numbers $(a_n > 0, \quad n \geq 1)$ and $(b_n,\ n \geq 1)$ such that

$$\lim_{n\to\infty} \mathbb{P}(X_{n,n} \leq a_n x + b_n) = H(x), \quad \forall x \in \mathbb{R}. \tag{1.1}$$

It is then said that F is in the domain of attraction of H, denoted by F$\in D(H)$. It is well-known that H is necessarily of these three types :

$$H(x) = \varphi_\alpha(x) = \exp(-x^{-\alpha})\ \mathbb{I}_{(x\geq 0)}\ ; \quad (type\ I) \tag{1.2}$$

$$H(x) = \Lambda(x) = \exp(-e^{-x})\ ; \qquad (type\ II) \tag{1.3}$$

$$H(x) = \psi_\gamma(x) = \exp(-(-x)^\gamma)\ \mathbb{I}_{(x\leq 0)} + \mathbb{I}_{(x>0)}\ ;\ \gamma > 0\ \ ; \ \ (type\ III) \tag{1.4}$$

where $\mathbb{I}_A$ is indicator function of the set A. It should be noticed that this function H represents an equivalence class of the following binary relation between the distribution functions :

$$H_1\mathcal{R}\ H_2\ \Leftrightarrow (\exists\ (a,b)\in \mathbb{R}_+^{*}\times\mathbb{R}, \forall\ x\in\mathbb{R},\ H_1(x)=H_2(ax+b)) \tag{1.5}$$

In this case, any of the function H in (1.2), (1.3) and (1.4) may be represented as

$$H(x)=H_\gamma(x)=\exp(-(1+\gamma x)^{-1/\gamma}),\ \text{pour } 1+\gamma x>0,$$

where $(1+\gamma x)^{-1/\gamma}$ is interpreted as e^{-x}. The values $\gamma>0$, $\gamma=0$ *et* $\gamma<0$ are respectively related to the types I, II et III.

In this paper, the class of the whole domain of attraction (the set of all distribution functions attracted to some H) is denoted by $\mathfrak{D}=\cup_{\alpha>0}\cup_{\gamma>0}D(\varphi_\alpha)\cup D(\psi_\gamma)\cup D(\Lambda)$. *In the sequel we shall use the domain attaction of* $G(\cdot)=F(e^{\cdot})$. When F$\in\mathfrak{D}$, G is in $\cup_{\gamma>0}\ D(\varphi_\alpha)\cup\ D(\psi_\gamma)$. It will be seen that any property of formula true for F$\in D(\psi_\gamma)$ ($\Longrightarrow G\in D(\psi_\gamma)$) is also true for F$\in\cup_{\gamma>0}\ D(\varphi_\alpha)\cup D(\psi_\gamma)$ ($\Longrightarrow G\in D(\Lambda)$) by letting $\gamma\to\infty$ (except in (3.52) below). This allows us to denote D$(\Lambda)=D(\psi_{+\infty})$. Consequently, any formula with the parameter $\gamma>0$ is true in the domain $D(\psi_\gamma)$ and the corresponding result for the domain $U_{\alpha>0}$ $D(\varphi_\alpha)\cup D(\Lambda)$ is obtained by letting $\gamma\to+\infty$, unless the contrary is specified.

The interested reader is referred to [4], [8] or [14] for an general introduction to extereme value theory. The characterization of these three domains received very much consideration in th two past decades. In the same time, many empirical discrimination results within the extremal domain are available. First, Mason [13] gave a complete characterization of the type II with the Hill statistic [10]. Tiago de Oliveira [17], Smith [15], Dekkers and al. [6], Hasofer et Wang [9] also proposed statistical tests for the selection between these three types. In the same spirit, Lo [11] introduced a class of nine statistics which characterizes completely all the extremal domain. Next, Diop et Lo [7] introduced the following family, depending on the continuous parameter τ, $0<\tau<\infty$,

$$T_n(\tau)=k^{-\tau}\sum_{j=\ell+1}^{j=k} i^{\tau}\left(\log\quad X_{n-j+1,n}-\log X_{n-j,n}\right), \tag{1.6}$$

where l, k and n are integers such that

$$0<l<k<n\ \geq 1,\quad l/k\to 0 \text{ and } \ k/n\to 0 \text{ as } n\to\infty, \tag{K}$$

where $X_{1,n}\leq ......\leq X_{n,n}$ are the order statistics based on $X_1,......X_n$. They established that for any couple (τ,ρ) with $0<\tau$ and $0<\rho$, the limiting values of $(T_n(\tau),\ T_n(\tau)\ /\ T_n(\rho))$ entirely discriminate the three types (see Remark **??** below).

We aim to characterize here the asymptotic normality of this family of statistics. In this paper, we shall focus the asymptotic normality of their finite-distributions. Statistical tests will be derived and simulations given. The parameter τ is used to obtain very sensitive and efficient results.

Before we go any further, let us give once some natotions. $G^{-1}(\cdot.)=\inf\{x,\ G(x)\geq\cdot\}$ is the generalized inverse of $G(\cdot)=F(e^{\cdot})$, the distribution function of Y=$logX$, for F$\in\mathfrak{D}$. We will have the following representations. We have, first for $F\in D(\varphi_\gamma)$,

$$G^{-1}(1-u)=\log c(1+f(u))-(\log u)/\gamma+\int_u^1 b(t)t^{-1}dt,\quad 0<u<1, \tag{1.7}$$

next for $F \in D(\psi_\gamma)$, $y_0 = \sup\{x, G(x) < 1\} < +\infty$

$$y_0 - G^{-1}(1-u) = c(1+f(u))\ u^{1/\gamma} \exp(\int_u^1 b(t)t^{-1}dt), 0 < u < 1 \tag{1.8}$$

and finally, for $F \in D(\Lambda)$,

$$G^{-1}(1-u) = d - s(u) + \int_u^1 s(t)t^{-1}dt), 0 < u < 1, \tag{1.9}$$

where $s(u) = c(1+f(u))\exp(\int_u^1 b(t)t^{-1}dt), 0 < u < 1$. In each of these formulae, (f(u), b(u))→ (0,0) when u→ 0. (1.7) and (1.8) are the Karamata representations while (1.9) is that of de Haan-Mason-Deheuvels .

2. RESULTS

In the sequel, we also need the following functions for $0 < \tau, \rho$:

$$\sigma_1^2(\tau) = 2\tau^2(\gamma+1)^2/\{(\gamma\tau+1)(\gamma(2\tau-1)+2)\} \tag{2.1}$$

$$\sigma_2^2(\tau) = (\gamma+1)^2(\tau\gamma^2-\gamma+2)/\{\gamma^2(\gamma\tau+1)(\gamma(2\tau-1)+2)\} \tag{2.2}$$

$$\sigma_3^2(\tau,\rho) = 2\left(\frac{\gamma\rho+1}{\gamma\tau+1}\right)^2\left\{\frac{\tau^2(\gamma\tau+1)}{\gamma(2\tau-1)+2} + \frac{\rho^2(\gamma\rho+1)}{\gamma(2\rho-1)+2} - \tau\rho\frac{\gamma(\rho+\tau)+2}{\gamma(\rho+\tau-1)+2}\right\}, \tag{2.3}$$

$$\sigma_4^2(\tau,\rho) = \gamma^{-2}\left(\frac{\gamma\rho+1}{\gamma\tau+1}\right)^2\left\{\frac{(\tau\gamma^2-\gamma+2)(\gamma\tau+1)}{\gamma(2\tau-1)+2} + \frac{(\rho\gamma^2-\gamma+2)(\gamma\rho+1)}{\gamma(2\rho-1)+2}\right.$$
$$\left. -2\frac{\gamma^3\tau\rho+\gamma(\tau+\rho-1)+2}{\gamma(\rho+\tau-1)+2}\right\} \tag{2.4}$$

$$\Gamma_1(\tau,\rho) = \frac{\tau\rho(\gamma+1)^2(\gamma(\tau+\rho)+2)}{(\gamma(\tau+\rho-1)+2)(\gamma\tau+1)(\gamma\rho+1)}, \tag{2.5}$$

$$\Gamma_2(\tau,\rho) = \frac{(\gamma+1)^2(\gamma^3\tau\rho+\gamma(\tau+\rho-1)+2)}{\gamma^2(\gamma(\tau+\rho-1)+2)(\gamma\tau+1)(\gamma\rho+1)} \tag{2.6}$$

Troughout the paper, we use the representations of the studied random variables $\log X_i,\ i \geq 1$, by $G^{-1}(1-U_i),\ i \geq 1$,where $U_1, U_2, ...$ is a sequence of independent random variables uniformly distributed on $(0,1)$. It follows that :

$$\{\log\ X_{n-i+1,n}, 1 \leq i \leq n\ ;\ n \geq 1\} \overset{d}{=} \left\{G^{-1}(1-U_{i,n}), 1 \leq i \leq n\ ;\ n \geq 1\right\} \tag{2.7}$$

where $U_{1,n} < ... < U_{n,n}$ are the order statistics based on $U_1, ..., U_n$. Here are the final notations

$$\widetilde{x}_n = G^{-1}(1-U_{k+1,n}),\ \widetilde{z}_n = G^{-1}(1-U_{l,n}), \tag{2.8}$$

$$x_n = G^{-1}(1-k/n),\ z_n = G^{-1}(1-l/n), \tag{2.9}$$

$$\widetilde{\mu}_n(\tau) = (n/k)^{\tau} \int_{\widetilde{x}_n}^{\widetilde{z}_n} (1 - G(t))^{\tau} dt; \ \mu_n(\tau) = (n/k)^{\tau} \int_{x_n}^{z_n} (1 - G(t))^{\tau} dt, \tag{2.10}$$

$$r(\tau, x, z) = (1 - G(x))^{-\tau} \int_{x}^{z} (1 - G(t))\tau \ dt, \tag{2.11}$$

$$r(\tau, x) \equiv r(\tau, x, y_0), \ r(x) \equiv r(1, x, y_0)$$

with x$\leq z \leq y_0 = \sup\{x, \ G(x) < 1\}$ and $\tau > 0$.

We are now able to give our results for these statistical processes for $0 < T < \infty$,

$$\{\mathbb{S}_n, n > 1\} = \left\{ k^{1/2}(T_n(\tau) - \widetilde{\mu}_n(\tau))/r(x_n), 0 < \tau < T, n > 1 \right\} \tag{2.12}$$

and

$$\{\mathbb{W}_n, n > 1\} = \left\{ k^{1/2}(T_n(\tau) - \mu_n(\tau))/r(x_n), 0 < \tau < T, n > 1 \right\} \tag{2.13}$$

Theorem 1. *Assume that (K) holds and let $F \in \mathfrak{D}$, $\tau > 1/2$ and $\rho > 1/2$. Then*

$$(\mathbb{S}_n(\tau), \ \mathbb{S}_n(\rho)) \qquad \rightarrow_d \ \mathfrak{N}_2(0, \begin{bmatrix} \sigma_1^2(\tau) & \Gamma_1(\tau, \rho) \\ \Gamma_1(\tau, \rho) & \sigma_1^2(\rho) \end{bmatrix}) \tag{2.14}$$

and

$$k^{1/2}(T_n(\tau)/T_n(\rho) - \widetilde{\mu}_n(\tau)/\widetilde{\mu}_n(\rho)) \qquad \rightarrow_d \quad \mathfrak{N}(0, \sigma_3^2(\tau, \rho)) \tag{2.15}$$

The centering coefficients are random in this theorem. In order to be able to replace them by non random ones, we need the following regularity condition

$$\gamma_n(k) = n^{1/2}\{f(U_{k,n}) - f(k/n)\} \rightarrow_p 0 \text{ as n} \rightarrow \infty \tag{RC}$$

Precisely, we have :

Theorem 2. *Assume that (K) holds and let $F \in \mathfrak{D}$, $\tau > 1/2$ and $\rho > 1/2$. Then,*

(1) $W_n(\tau) \rightarrow_d \ \mathfrak{N}(0, \sigma_3^2(\tau))$ *if and only if (RC) holds.*
(2) *Moreover, whenever (RC) holds,*

$$(\mathbb{W}_n(\tau), \ \ \mathbb{W}_n(\rho)) \ \rightarrow_d \ \mathfrak{N}_2(0, \begin{bmatrix} \sigma_2^2(\tau) & \Gamma_2(\tau, \rho) \\ \Gamma_2(\tau, \rho) & \sigma_2^2(\rho) \end{bmatrix}) \tag{2.16}$$

and

$$k^{1/2}(T_n(\tau)/T_n(\rho) - \mu_n(\tau)/\mu_n(\rho)) \qquad \rightarrow_d \quad \mathfrak{N}(0, \sigma_4^2(\tau)) \tag{2.17}$$

Remark 1. *According to Theorem 1 and Lemma 1 below, the couple ($\mathbb{S}_n(\tau)$, $\mathbb{S}_n(\rho)/\mathbb{S}_n(\tau)$) tends in probability to (0, ρ/τ), ($1/(\tau\gamma)$, ρ/τ) or (0, $(\rho\tau{+}1)/(\gamma\tau{+}1)$) according to the three cases F$\in D(\Lambda)$, F$\in D(\varphi_\gamma)$ or F$\in D(\psi_\gamma)$. These results already obtained in* [7] *in almost sure limits, cleary give a total discrimation of the whole domain of attraction. The theorems then propose statistical tests based of the consistency results.*

Remark 2. *It is important to see that the variances given in (2.1)-(2.4) are defined only for $\tau > 1/2$ and $\rho > 1/2$.*

3. PROOFS OF THE RESULTS.

We need these two lemmas. First, define for $-\infty < x < z < y_0,\ 0 < \tau, \rho :$

$$w\left(\tau,\rho,x,z\right)=\frac{1}{\left(1-G(x)\right)^{\tau+\rho}}\int_x^z\int_y^z\left(1-G(t)\right)^{\tau}\left(1-G(y)\right)^{\rho}dt\ dy, \tag{3.1}$$

with $w\left(\tau,\rho,x,y_0\right)\equiv w\left(\tau,\rho,x\right).$

Lemma 1. *Let* $\tau>0,\ \rho>0,\ \gamma>0.$

(1) If $F\in D(\Lambda)\mathbb{U}D(\Phi)$, then $1-(1-G)^{\tau}\in D(\Lambda)$ and $r\left(\tau,G^{-1}(1-u)\right)$ is slowly varying at zero and

$$w\left(\tau,\rho,x,z\right)\sim w\left(\tau,\rho,x\right)\sim r(\tau,x)r(\tau+\rho,x) \tag{3.2}$$

as $x\rightarrow y_0,\ \ z\rightarrow y_0$ and $\frac{1-G(z)}{1-G(x)}\rightarrow 0.$

(2) If $F\in D(\Psi_{\gamma})$, then $1-(1-G)^{\tau}\epsilon\ D(\Psi_{\gamma\tau})$ and $r\left(\tau,G^{-1}(1-u)\right)$ is regularly varying at zero with exponent γ^{-1}. Moreover

$$w\left(\tau,\rho,x,z\right)\sim w\left(\tau,\rho,x\right)\sim e(\gamma,\tau,\rho)r(\tau,x)r(\tau+\rho,x), \tag{3.3}$$

as $x\rightarrow y_0,\ \ z\rightarrow y_0,\ \frac{1-G(z)}{1-G(x)}\rightarrow 0$, where $e(\gamma,\tau,\rho)=\frac{\gamma(\tau+\rho)+1}{\gamma(\tau+\rho)+2}$.

(3) If $F\in\mathfrak{D}$, then $\left(1-G(G^{-1}(1-u))\right)/u\rightarrow 1$ as $u\rightarrow 0$.

(4) If F$\in\mathfrak{D}$, r$(\tau,x)/r(\rho,x)\rightarrow(\gamma\rho+1)/(\gamma\tau+1)$ as x$\rightarrow y_0$, for $0<\tau,\rho$

Proof. **Point (1)**. Let $F\in D(\Lambda)\mathbb{U}D(\Phi)$. Then $G\in D(\Lambda)$ (see [11], for instance). This is equivalent to

$$\forall t>0\in\mathbb{R},\lim_{x\rightarrow y_0}\frac{1-G(x+t(x)r(x))}{1-G(x)}\rightarrow e^{-t},\ \text{as t(x)}\rightarrow t\text{ as }x\rightarrow y_0, \tag{3.4}$$

(see Lemma 2.5.1 in [4], for instance). Thus

$$\forall t>0\in\mathbb{R},\lim_{x\rightarrow y_0}\frac{(1-G(x+t(x)r(x)/\tau))^{\tau}}{(1-G(x))^{\tau}}\rightarrow e^{-t},\ \text{as t(x)}\rightarrow t\text{ as }x\rightarrow y_0, \tag{3.5}$$

The just cited lemma implies that 1-(1-G)$^{\tau}\in$ D(Λ) with, as $x\rightarrow y_0$,

$$r_{\tau}(x)=r(\tau,x)=\left(1-G(x)\right)^{-\tau}\int_x^{y_0}\left(1-G(t)\right)^{\tau}dt\sim r(x)/\tau. \tag{3.6}$$

Since r(G$^{-1}(1-u)$) is slowly varying at zero for G$\in D(\Lambda)$ by Lemma 3.4 in [11], the same is true for $r(\tau,G^{-1}(1-u))$. It remains to prove (3.2). Put for $\rho>0$ and $\tau>0$, $1-G^{*}(x)=\left(1-G(x)\right)^{\rho}\int_x^{y_0}\left(1-G(t)\right)^{\tau}dt$. It is clear that 1-G* is the tail of a distribution function. Further, by Lemma 3.3 in [11], t(x)=t r$(\tau+\rho,x)/r(x)\rightarrow t/(\tau+\rho)$ as $x\rightarrow y_0$. Next, since $G\in D(\Lambda)$, this, (3.4) and the Hospital rule together yield, $x\rightarrow y_0$,

$$\left(\int_{x+t(x)\ r(x)}^{y_0}(1-G(t))^{\tau}dt\right)/\left(\int_x^{y_0}(1-G(t))^{\tau}dt\right)\rightarrow e^{-\frac{t\tau}{\rho+\tau}}. \tag{3.7}$$

All what preceeds enables to get

$$\frac{1-G^{*}\left(x+tr(\tau+\rho,x)\right)}{1-G^{*}(x)}=\frac{\left(1-G(x+x(t)r(x)\right)^{\rho}}{1-G(x)} \tag{3.8}$$

$$\times\left(\int_{x+t(x)\ r(1,x)}^{y_0}(1-G(t))^{\tau}dt\right)/\left(\int_{x}^{y_0}(1-G(t))^{\tau}dt\right)$$

$$\rightarrow e^{-\frac{t\rho}{\rho+\tau}}e^{-\frac{t\rho}{\rho+\tau}}=e^{-t}. \tag{3.9}$$

Finally the same Lemma 2.5.1 in [4] implies that $G^{*}\in D(\Lambda)$ and, as $x\rightarrow y_0$,

$$r(\tau+\rho,x)\sim\frac{1}{1-G^{*}(x)}\int_{x}^{y_0}(1-G^{*}(t))\,dt=r(\tau,\rho,x)/r(\tau,x). \tag{3.10}$$

This proves (3.4). To complete the proof of point (1) of the lemma, it remains show that

$$w(\tau,\rho,x,z)\sim w(\tau,\rho,x), \tag{3.11}$$

as $z\wedge x\uparrow y_0$ whenever (1-G(z))/(1-G(x)) as $z\wedge x\uparrow y_0$. Indeed we have

$$w(\tau,\rho,x,z)=\frac{1}{(1-G(x))^{\tau+\rho}}\int_{x}^{z}\int_{y}^{z}(1-G(t))^{\tau}(1-G(y))^{\rho}dt\ dy \tag{3.12}$$

$$=\frac{1}{(1-G(x))^{\tau+\rho}}\left[\int_{x}^{y_0}\int_{y}^{y_0}.-\int_{x}^{z}\int_{z}^{y_0}.-\int_{z}^{y_0}\int_{y}^{y_0}.\right]\equiv w(\tau,\rho,x)(1-E_2-E_3)$$

with $E_2 =$

$$\left(\int_{x}^{z}\int_{z}^{y_0}.\,(1-G(t))^{\tau}\,(1-G(y))^{\rho}\,dt\ dy\right)/\left(\int_{x}^{y_0}\int_{y}^{y_0}(1-G(t))^{\tau}\,(1-G(y))^{\rho}\,dt\ dy\right)$$

$$=\frac{(1-G(x))^{\tau}(1-G(z))^{\rho}r(\tau,x)r(\rho,z)}{(1-G(x))^{\tau+\rho}w(\tau,\rho,x)}.$$

By (3.10),

$$E_2\sim(\frac{1-G(z)}{1-G(x)})^{\rho}\frac{r(\tau,x)r(\rho,z)}{r(\tau,x)r(\tau+\rho,x)}\sim\frac{\tau+\rho}{\tau}(\frac{1-G(z)}{1-G(x)})^{\rho}\frac{r(z)}{r(x)} \tag{3.13}$$

But

$$\frac{r(z)}{r(x)}\leq\frac{r(z)}{x-z}\longrightarrow 0, \tag{3.14}$$

whenever (1-G(z))/(1-G(x))$\longrightarrow 0$ as $z\wedge x\uparrow y_0$, by Lemma 4.2 in [11] for all F$\in\mathfrak{D}$. (3.13) and (3.14) together yield that $E_2\longrightarrow 0$. By the same arguments, $E_3\leq$

$$\left(\int_{z}^{y_0}\int_{y}^{y_0}(1-G(t))^{\tau}\,(1-G(y))^{\rho}\,dt\ dy\right)/\left(\int_{x}^{z}\int_{y}^{z}(1-G(t))^{\tau}\,(1-G(y))^{\rho}\,dt\ dy\right)$$

$$=2\frac{w(\tau,\rho,z)}{(z-x)^2}\sim 2\frac{r(\tau+\rho,z)}{z-x}\ \frac{r(\tau,z)}{z-x}\sim\frac{2}{\tau(\tau+\rho)}(\frac{r(z)}{x-z})^{2}\longrightarrow 0.$$

This completes the proof of Point (1) of the Lemma.

Proof of Point (2). Let $F\in D(\Psi_{\gamma})$. Then $G\in D(\Psi_{\gamma})$. Use the Karamata representation of G:

$$1-G(t)=c(t)(y_0-t)^{\gamma}\exp\int_{1}^{\frac{1}{y_0-t}}\frac{b(t)}{t}dt, \tag{3.15}$$

with $b(t)\rightarrow 0$ et $c(t)\rightarrow c>0$. We get that for all $\tau>0$,

$$(1-G(t))^{\tau}=c(t)^{\tau}(y_0-t)^{\gamma\tau}\exp\int_{1}^{\frac{1}{y_0-t}}\frac{\tau b(t)}{t}dt. \tag{3.16}$$

This means that 1-(1-G)$^{\tau}$ $\in D(\Psi_{\tau\gamma})$. Now, one easily gets for any $\tau > 0$ and $\rho > 0$, as $x \rightarrow y_0$,

$$r(\tau, x) = \frac{(y_o - x)}{(\gamma\tau + 1)}(1 + o(1)) \tag{3.17}$$

and

$$w(\tau, \rho, x) = \frac{(y_o - x)^2}{(\gamma\tau + 1)(\gamma(\tau + \rho) + 2)}(1 + o(1)). \tag{3.18}$$

But

$$r(\tau + \rho, x)\ r(\tau, x) = \frac{(y_o - x)^2}{(\gamma\tau + 1)(\gamma(\tau + \rho) + 1)}(1 + o(1)) \tag{3.19}$$

Comparing these two last formulae gives (3.3) for z=y$_0$. Remind that (3.11) has been proved for any G for F$\in \mathfrak{D}$. Then (3.3) is completely proved. To finish, recall that one has, by vertue of (3.17)

$$r(\tau, G^{-1}(1-u)) \sim \frac{y_o - G^{-1}(1-u)}{(\gamma\tau + 1)},$$

which is regularly varying at zero with exponent $1/\gamma$ by the classical Karamata representation of G$\in D(\Psi_{\gamma})$ (derived from (3.15)). The proof of Point (2) of the lemma is finished.
Point (3).This is Lemma 3.2 in [11].
Point (4). This is Lemma 3.3 in [11]. □

Lemma 2. *Assume that (K1) holds. Then there exists a probability space carrying a sequence of independent random variables uniformly distributed on (0,1) denoted* U_1, U_2, ... *and a sequence of brownians bridges* $\{B_n(s), 0 \leq s \leq 1\}$ $n = 1, 2, ...$*such that for all* $v, 0 \leq v < 1/4$ *and for all* $\tau > 0$,

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \frac{\sqrt{n}\left\{(U_n(s))^{\tau} - s^{\tau}\right\} - \tau s^{\tau-1} B_n(s)}{s^{\tau - 1 + 1/2 - v}} = O_p(n^{-\nu}). \tag{3.20}$$

where $U_n(.)$ is the empirical distribution function based on $U_1, U_2, ...U_n$, $n \geq 1$.

Proof. Csorgö, $Csörgö$, $Horvàth$ and $Mason$(1986) have constructed a probability space on wich the two following approximations hold simultaneously for 0< $\nu < 1/4$, (voir [16], p.501).

$$\sup_{U_{1,n} \leq \text{ s } \leq U_{n,n}} \frac{\sqrt{n}\left[U_n(s) - s\right] - B_n(s)}{(s(1-s))^{1/2 - v}} = O_p(n^{-v}) \tag{3.21}$$

et

$$\sup_{1/n \leq \text{ s } \leq 1 - 1/n} \frac{\sqrt{n}\left[s - V_n(s)\right] - B_n(s)}{(s(1-s))^{1/2 - v}} = O_p(n^{-v}), \tag{3.22}$$

where V$_n$ is the empirical uniform quantile function. Now, by using twice the mean value theorem, we get

$$\sqrt{n}\left\{(U_n(s))^{\tau} - s^{\tau}\right\} - \tau s^{\tau-1} B_n(s) = \tau s^{\tau-1} \alpha_n(s) + \tau(\tau-1)\ \alpha_n(s)(\zeta_n'(s) - s)(\zeta_n''(s))^{\tau-2}$$

$$\equiv R_n(1, s) + R_n(2, s), \tag{3.23}$$

where $\alpha_n(\cdot)$ is the empirical uniform process and $\{\zeta_n'(s), \zeta_n"(s)\} \subset [U_n' s) \wedge s, U_n(s) \vee s]$. It is immediate in view of (3.21) that,

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \frac{|R_n(1,s)|}{s^{\tau-1+1/2-v}} = O_p(\tau n^{-\nu}). \tag{3.24}$$

Furthermore, by Lemma 13 in [2]

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \sup_{U_n(s) \wedge s \leq b_n(s) \leq U_n(s) \wedge s} \left|s^{-1} b_n(s)\right| = \widetilde{O}_p(1), \tag{3.25}$$

with the notation : $f = \widetilde{O}_p(1) \Leftrightarrow (f = \widetilde{O}_p(1)$ and $\frac{1}{f} = O_p(1))$. Remark also that (3.19) is also true for $\nu = 0$. Thus

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \left|\alpha_n(s)/\sqrt{s}\right| \leq O_p(1) + \sup_{0<s<1} \left|B_n(s)/\sqrt{s}\right| = O_p(1)$$

Hence,

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \frac{R_n(2,s)}{s^{\tau-1}} = \tau(\tau-1) \sup_{U_{1,n} \leq s \leq U_{n,n}} n^{-1/2} \left|\frac{\alpha_n(s)^2}{s}\right| \sup_{U_{1,n} \leq s \leq U_{n,n}} \left|\zeta_n"(s)/s\right|^{\tau-2}$$

$$= O_p(\tau(\tau-1)n^{-1/2}). \tag{3.26}$$

Thus

$$\sup_{U_{1,n} \leq s \leq U_{n,n}} \frac{|R_n(2,s)|}{s^{\tau-1}} = O_p(n^{-\nu}). \tag{3.27}$$

This completes the proof of the lemma. □

For convenience, we use in the sequel these notations, for $\tau > 0$ and $0 < s \leq 1$:

$$\widetilde{\alpha}_n(\tau, s) = \sqrt{n}\left\{(U_n(s))^{\tau} - s^{\tau}\right\} \tag{3.28}$$

and

$$\widetilde{B}_n(\tau, s) = \tau s^{\tau-1} B_n(s) \tag{3.29}$$

We are now able to prove our results.

Proof. of Theorem 1.

It is easily seen that

$$T_n(\tau) = (\frac{n}{k})^{\tau} \int_{\widetilde{x}_n}^{\widetilde{z}_n} (1 - G_n(s))^{\tau} \, ds \tag{3.30}$$

and then

$$\frac{k^{1/2}}{r(x_n)} \left\{T_n(\tau) - \widetilde{\mu}_n(\tau)\right\} = N_n(1,\tau) + R_n(1) + R_n(2) + R_n(3), \tag{3.31}$$

where the terms $\mathrm{R}_n(j)$ are specified and handled in the coming claims.

Claim 1. $R_n(3) = \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{\widetilde{z}_n} \widetilde{\alpha}_n(\tau, 1-G(t)) - \widetilde{B}_n(\tau, 1-G(t)) \, dt \to_p 0.$

□

Proof. For $\widetilde{x}_n < t < \widetilde{z}_n$, we have $U_{1,n} < 1 - G(t) \leq U_{n,n}$ so that by Lemma 2

$$R_n(3) = O_p(n^{-\nu}) \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{\widetilde{z}_n} (1-G(t))^{\tau-\frac{1}{2}-\nu}\ dt \tag{3.32}$$

For $\tau > 1/2$, we chose $\nu > 0$ so that $\tau - \frac{1}{2} - \nu > 0$. By arguments given in Claim 2, we have $(nU_{k,n}/k) \to_p 1$. This and Point (3) of Lemma 1, together give

$$|R_n(3)| \leq O_p(n^{-\nu}) \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{y_0} (1-G(t))^{\tau-\frac{1}{2}-\nu} dt \tag{3.33}$$

$$= O_p(n^{-\nu})(1-G(\widetilde{x}_n))^{-\nu} \frac{r(\tau-\frac{1}{2}-\nu, \widetilde{x}_n)}{r(1,x_n)} = O_p(k^{-\nu}) \tag{3.34}$$

because of points (3) and (4) of lemma 1.

Claim 2. $R_n(1) = \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{\widetilde{x}_n}^{x_n} \widetilde{B}_n(\tau, 1-G(t))\ dt \to_p 0.$

□

Proof. For any integer d fixed with $1 < d < n$, $U_{d,n}$ has the same law that of S_d/S_{n+1}, where S_n is the partial sum of the n first elements of a sequence of independent random variables with standard exponential distribution. (see [16], Proposition 1, p.335). Hence, by the law of large numbers, $(nU_{k,n}/k) \to_p 1$ for k$\to \infty$, while for l fixed, $(nU_{l+1,n}/l) = \widetilde{O}_p(1)$ as n$\to \infty$. This implies that for any $\epsilon > 0$,

$$\forall \lambda > 1 \quad (\exists n_1 > 0)\ \ \forall n \geq n_1, \ \ k/n\lambda \leq U_{k+1,n} \leq \lambda k/n \tag{3.35}$$

holds with probability at least greater that 1-ϵ (denoted w.p.g.1-ϵ) and

$$(\exists n_2 > 0)\ \ (\exists \lambda > 0),\ (\forall n \geq n_2, \ \ l/n\lambda \leq U_{l+1,n} \leq \lambda l/n \tag{3.36}$$

w.p.g.1-ϵ. We also use the notation $\mathrm{t}_n(\cdot) = \cdot/n$. Hence, for $t > (\widetilde{x}_n \wedge k/n)$,

$$1 - G(t) \leq (U_{k,n} \wedge k/n) \leq k/(\lambda n),$$

w.p.g.1-ϵ, for large n$\geq n_1$. Thus, w.p.g.1-ϵ, for n$\geq n_1$

$$\mathbb{E}\,|R_n(1)| \leq 2\tau \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{t_n(k/\lambda)}^{t_n(k)} (1-G(t))^{\tau-1}(G(t)(1-G(t)))^{1/2}\ dt \tag{3.37}$$

$$= o(1) \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \Big\{ (1-G(t_n(k/\lambda))^{\tau-1/2} r(\tau-1/2, t_n(k/\lambda))$$

$$-(1-G(t_n(k))^{\tau-1/2} r(\tau-1/2, t_n(k)) \Big\}$$

$$\to (\frac{1}{(\gamma(\tau-1/2)+1)\lambda^{\tau-1/2+1/\gamma}} - \frac{1}{(\gamma(\tau-1/2)+1)}),$$

because of Points (1) and (2) of Lemma 1 and this for any $\lambda > 1$. Then, as $\lambda \downarrow 1$, this term is zero. From there, a probabilistic argument can show that $\mathbb{E}\,|R_n(1)| \to 0$ and then $R_n(1) \to_p 0$ by the Markov inequality.

Claim 3. $\frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{z_n}^{\widetilde{z}_n} \widetilde{B}_n(\tau, 1-G(t))\ dt \to_p 0.$

□

Proof. By using (3.36) and the same methods in what preceeds, we get, w.p.g.1-ϵ, for n$\geq n_2$ *and* some $\lambda > 0$,

$$\mathbb{E}\left|R_n(2)\right| \leq 2\tau \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{t_n(l/\lambda)}^{t_n(l)} (1-G(t))^{\tau-1}(G(t)(1-G(t)))^{1/2}\ dt \tag{3.38}$$

$$= o(1)(l/k)^{\tau-1/2} \frac{r(1,z_n)}{r(1,x_n)} \frac{(l/n)^{\tau-1/2}}{r(1,z_n)} \Big\{ (1-G(t_n(l/\lambda))^{\tau-1/2} r(\tau-1/2, t_n(l/\lambda))$$
$$(1-G(t_n(l))^{\tau-1/2} r(\tau-1/2, t_n(l)) \Big\}.$$

But the same methods used juste below show that

$$\frac{(n/l)^{\tau-1/2}}{r(1,z_n)} \Big\{ (1-G(t_n(l/\lambda))^{\tau-1/2} r(\tau-1/2, t_n(l/\lambda)) - (1-G(t_n(l))^{\tau-1/2} r(\tau-1/2, t_n(l)) \Big\}$$

converges to $\frac{1}{(\gamma(\tau-1/2)+1)\lambda^{\tau-1/2+1/\gamma}} - \frac{1}{(\gamma(\tau-1/2)+1)}$ and $(l/k)^{\tau-1/2} \frac{r(1,z_n)}{r(1,x_n)} \to 0$ in view of points (1) and (2) of Lemma 1. Hence $\mathbb{E}\left|R_n(2)\right| \to 0$ and *thus* $R_n(2) \to_p 0$.

Claim 4. $N_n(1,\tau) = \frac{(n/k)^{\tau-1/2}}{r(1,x_n)} \int_{x_n}^{z_n} \widetilde{B}_n(\tau, 1-G(t))\ dt \sim \mathfrak{N}(0,\sigma_1^2(\tau))$.

It is clear that $N_n(1,\tau)$ is a centered normal random variable with variance

$$\sigma_{1,n}^2(\tau) = \frac{(n/k)^{2\tau-1}}{r(1,x_n)^2} \tau^2 \int_{x_n}^{z_n} \int_{x_n}^{z_n} (1-G(t))^{\tau-1}(1-G(s))^{\tau-1} \times$$
$$\{\min((1-G(t)),(1-G(t))) - (1-G(t))(1-G(s)\}\ ds\ dt \tag{3.39}$$

We now cut this into

$$\sigma_{1,n}^2(\tau) = \frac{(n/k)^{2\tau-1}}{r(1,x_n)^2} \tau^2 \left[\int_{x_n}^{z_n} \int_{x_n}^{s} . \quad + \int_{x_n}^{z_n} \int_{s}^{z_n} \right] =: \sigma_{1,n,1}^2 + \sigma_{1,n,2}^2\ , \tag{3.40}$$

where

$$\sigma_{1,n,1}^2(\tau) = \frac{(n/k)^{2\tau-1}}{r(1,x_n)^2} \tau^2 \int_{x_n}^{z_n} \int_{x_n}^{s} G(t)(1-G(t))^{\tau-1}(1-G(s))^{\tau} ds\ dt \tag{3.41}$$

and

$$\sigma_{1,n,2}^2(\tau) = \frac{(n/k)^{2\tau-1}}{r(1,x_n)^2} \tau^2 \int_{x_n}^{z_n} \int_{s}^{z_n} G(s)(1-G(t))^{\tau}(1-G(s))^{\tau-1} ds\ dt. \tag{3.42}$$

Points (1), (2) and (3) of Lemma 1 enable to get

$$\sigma_{1,n,1}^2(\tau) \sim \frac{\tau^2(\gamma+1)^2}{(\gamma(2\tau-1)+2)(\gamma\tau+1)}. \tag{3.43}$$

Next, an integration by parts in $\sigma_{1,n,1}^2(\tau)$ with $u = \int_{x_n}^{s} (1-G(t))^{\tau-1} dt$ and $dv = d\left(\int_{s}^{z_n} (1-G(t))^{\tau} dt\right)$, permits to have $\sigma_{1,n}^2(2) = \sigma_{1,n}^2(1)(1+o(1))$. And this leads to :

$$\sigma_{1,n}^2(\tau) \to \sigma_1^2(\tau) \text{ as } n \to \infty.$$

This establishes the asymptotic normality of the margins of $\mathbb{S}_n(\tau)$. Moreover, the finite distributions are obtained from that of the brownian bridge. They inherit its multivariate normal distribution. Its remains to compute the covariance function, that is the covariance between $\mathbb{S}_n(\tau)$ and $\mathbb{S}_n(\rho)$, given by the limit of

$$\mathbb{E}(N_n(1,\tau)N_n(1,\rho)) = \frac{\tau\rho(n/k)^{\tau+\rho-1}}{r(1,x_n)^2} \times \tag{3.44}$$

$$\int_{x_n}^{z_n}\int_{x_n}^{z_n}(1-G(s))^{\tau-1}(1-G(s))^{\rho-1}\left[(1-G(t))\Lambda(1-G(s))-(1-G(t))(1-G(s)\right]ds\ dt$$

The same techniques used in (3.40), cutting and integration by parts, yield

$$E(N_n(1,\tau)N_n(1,\rho)) = \tag{3.45}$$

$$\frac{\tau\rho e_3(\gamma)r(\tau,x_n)r(\tau+\rho-1,x_n)}{r(1,x_n)^2}(1+o(1))+\frac{\tau\rho e_3(\gamma)r(\rho,x_n)r(\tau+\rho-1,x_n)}{r(1,x_n)^2}(1+o(1))$$

with $e_3(\gamma)=\frac{\gamma(\tau+\rho-1)+1}{\gamma(\tau+\rho-1)+2}$. The limit of this is $\Gamma_1(\tau,\rho)$ by virtue of Lemma 1. This achieves the proof of (2.14). It remains to prove (2.15). By (2.14) and Lemma 1, we have the following one term expansion for any $\tau>1/2$,

$$\begin{aligned} T_n(\tau) &= \widetilde{\mu}_n(\tau)+\frac{N(1,\tau)r(x_n)}{k^{1/2}}+o_p(r(x_n)k^{-1/2}) \\ &= \widetilde{\mu}_n(\tau)(1+\frac{N(1,\tau)(\gamma\tau+1)}{(\gamma+1)k^{1/2}}+o_p(k^{-1/2})). \end{aligned}\tag{3.46}$$

Then

$$T_n(\rho)^{-1}=\widetilde{\mu}_n(\rho)^{-1}(1-\frac{N(1,\rho)(\gamma\rho+1)}{(\gamma+1)k^{1/2}}+o_p(k^{-1/2}))$$

and, since by Lemma 1, $\widetilde{\mu}_n(\tau)\widetilde{\mu}_n(\rho)^{-1}\sim(\gamma\rho+1)/(\gamma\tau+1)$, then

$$T_n(\tau)/T_n(\rho)=\widetilde{\mu}_n(\tau)/\widetilde{\mu}_n(\rho)+\frac{(\gamma\rho+1)N(1,\tau)}{(\gamma+1)k^{1/2}}-\frac{(\gamma\rho+1)^2}{(\gamma+1)(\gamma\tau+1)k^{1/2}}+o_p(k^{-1/2}).$$

This implies

$$\begin{aligned} \sqrt{k}(T_n(\tau)T_n(\rho)^{-1}-\widetilde{\mu}_n(\tau)\widetilde{\mu}_n(\rho)^{-1}) &= \frac{(\gamma\rho+1)N(1,\tau)}{(\gamma+1)}-\frac{(\gamma\rho+1)^2}{(\gamma+1)(\gamma\tau+1)}N(1,\rho)+o_p(1) \\ &=: N(4,\tau,\rho)+o_p(1). \end{aligned}\tag{3.47}$$

We finish by evaluating the variance of $N(4,\tau,\rho)$. But $\mathbb{E}(N(1,\tau)^2)=\sigma_1^2(\tau)$ and $\mathbb{E}(N(1,\tau)N(1,\rho))=\Gamma_1(\tau,\rho)$. From there, it is easy to show that $\mathbb{E}(N(4,\tau,\rho)^2)$ is

$$\sigma_3^2(\tau,\rho)=2\left(\frac{\gamma\rho+1}{\gamma\tau+1}\right)^2\left\{\frac{\tau^2(\gamma\tau+1)}{\gamma(2\tau-1)+2}+\frac{\rho^2(\gamma\rho+1)}{\gamma(2\rho-1)+2}-\tau\rho\frac{\gamma(\rho+\tau)+2}{\gamma(\rho+\tau-1)+2}\right\},$$

which is $\sigma_3^2(\tau,\rho)$. The proof of Theorem 1 is now complete. □

Proof. of Theorem 2.

From (3.31) and the different claims, we get for $\tau>1/2$,

$$\mathbb{W}(\tau)=N_n(1,\tau)+D_n+R_n(4)+o_p(1),\tag{3.48}$$

with $R_n(4)=\frac{\sqrt{k}}{r(1,x_n)}(n/k)^\tau\int_{z_n}^{\widetilde{z}_n}(1-G(t))^\tau dt$ and $D_n=\frac{\sqrt{k}}{r(1,x_n)}(n/k)^\tau\int_{\widetilde{x}_n}^{x_n}(1-G(t))^\tau dt$. By (3.36), for any $\epsilon>0$, for some $\lambda>0$ and for $n\geq n_2$,

$$|R_n(4)|\leq\frac{1}{k^{\tau-1/2}r(1,x_n)}(n/l)^\tau\int_{t_n(l)}^{t_n(\lambda l)}(1-G(t))^\tau dt\tag{3.49}$$

$$=\frac{(n/l)^\tau}{k^{\tau-1/2}r(1,x_n)}\left\{(1-G(t_n(l))^\tau\ r(\tau,t_n(l))-(1-G(t_n(\lambda l))^\tau\ r(\tau,t_n(\lambda l))\right\},$$

w.p.g.ϵ. The latter term goes to zero with $k^{-\tau+1/2}r(z_n)/r(x_n)$. This in turn gives

$$R_n(4)\to_p 0.$$

Next, by the fact that $nU_{k,n}/k \to_p 1$ whenever k$\to\infty$ with k/n$\to 0$, we have

$$D_n = (1+o_p(1))\frac{\sqrt{k}}{r(1,x_n)}(\widetilde{x}_n - x_n). \tag{3.50}$$

But, by Lemma 3.6 in [11] (Part II, p.17) and by using (3.22),

$$\frac{\sqrt{k}}{r(1,x_n)}(\widetilde{x}_n - x_n) = \tag{3.51}$$

$$-(1+o_p(1))\ e_1(\gamma)(n/k)^{1/2}B_n(k/n) + e_2(\gamma)k^{1/2}\gamma_n(k)(1+o_p(1)) + o_p(1).$$

where $e_1(\gamma) = (\gamma{+}1)/\gamma$, $e_2(\gamma) = \gamma\mathbb{I}_{(F\in D(\psi_\gamma))}{+}1$. Put N(2)$= -e_1(\gamma)(n/k)^{1/2}B_n(k/n)$ and $N(3,\tau) = N(1,\tau) + N(2)$. It is easy to see that $\mathbb{E}(N(2)^2) = (e_1(\gamma)^2)$. Further, the covariance between $N(1,\tau)$ and $N(2)$ is

$$-\frac{\tau(\gamma+1)}{r(1,x_n)\gamma}\left(\frac{n}{k}\right)^{1/2}\left(\frac{n}{k}\right)^{\tau-1/2}\int_{x_n}^{z_n}(1{-}G(t))^{\tau-1}\left[\min\left(\frac{k}{n},1-G(t)\right)-\frac{k}{n}(1-G(t))\right]dt.$$

$$\sim -\frac{\tau(\gamma+1)}{r(1,x_n)\gamma}\left(\frac{n}{k}\right)^{\tau}\int_{x_n}^{z_n}(1-G(t))^{\tau}dt \sim -\frac{\tau(\gamma+1)}{r(1,x_n)\gamma} \sim -\frac{\tau(\gamma+1)^2}{\gamma(\gamma\tau+1)}. \tag{3.52}$$

Hence, $N(3,\tau)$ is a normal random variable with variance is $\sigma_2^2(\tau)$. This, (3.49), (3.50) and (3.51) together establish the characterization given in Part 1 of Theorem 2. We have now the means to evaluate the covariance between $N_n(3,\tau)$ and $N_n(3,\rho))$ and then $\Gamma_2(\tau,\rho)$ which is

$$\Gamma_2(\tau,\rho) = \Gamma_1(\tau,\rho) - \frac{\gamma\tau(\gamma+1)^2}{\gamma^2(\gamma\tau+1)} - \frac{\tau\rho(\gamma+1)^2}{\gamma^2(\gamma\tau+1)} + (\frac{\gamma+1}{\gamma})^2.$$

It remains to get the law of $\mathrm{T}_n(\tau)/T_n(\rho)$ under (RC). The same arguments used in (3.47) yield

$$\sqrt{k}(T_n(\tau)T_n(\rho)^{-1}{-}\mu_n(\tau)\mu_n(\rho)^{-1}) = \frac{(\gamma\rho+1)N(3,\tau)}{(\gamma+1)}{-}\frac{(\gamma\rho+1)^2}{(\gamma+1)(\gamma\tau+1)}N(3,\rho){+}o_p(1)$$

$$=: N(5,\tau,\rho) + o_p(1). \tag{3.53}$$

An easy computation enables to get $\mathbb{E}(N(5,\tau,\rho)^2) =$

$$\gamma^{-2}\left(\frac{\gamma\rho+1}{\gamma\tau+1}\right)^2\left\{\frac{(\tau\gamma^2-\gamma+2)(\gamma\tau+1)}{\gamma(2\tau-1)+2} + \frac{(\rho\gamma^2-\gamma+2)(\gamma\rho+1)}{\gamma(2\rho-1)+2}\right. \tag{3.54}$$

$$\left. -2\frac{\gamma^3\tau\rho+\gamma(\tau+\rho-1)+2}{\gamma(\rho+\tau-1)+2}\right\} = \sigma_4^2(\tau,\rho).$$

This completely achieves the proofs of the theorems. □

4. SIMULATIONS ET APPLICATIONS

We make now simulations about the results. We choose one simple represent of each domain as follows. When the distribution depends on a parameter, several values are tested. Two hundred replications are used for each run.

Theorem 2 suggests to take the reject region as D=$\{|Z_n| \geq |z_0|\}$ where $Z_n = k^{1/2}(\mathbb{S}_n - coef - 0.5/k)$, z_0 is the observed Z_n, coef=$(\gamma\tau+1)/(\gamma\rho+1)$ and finally, 0.5/k is a correcting term. The p-value P=P($\{|Z_n| \geq |z_0|\}$), computed with the normal approximation, is reported for many values of the parameter τ, largely grater than 5%.

Type III with index β			Type II with index α			Type I		
$F(x) = 1 - (1-x)^{\beta}$			$F(x) = 1 - x^{-\alpha}$			$F(x) = 1 - e^{-x}$		
$\tau \backslash \beta = 1$	σ	P %	$\tau \backslash \alpha = 1$	σ	P %	$\tau \backslash$	σ	P %
10	1.007	37	1.5	.83	43	10	.67	53
15	.90	44	10	.67	53	15	.66	56
20	1.07	45	15	.72	57	20	.72	55

These very good simulations results suggest to the make the following recommandations for recognizing the right extremal domain.

(1) make satisfactory simulations with the sample size of one's data for suitable values of k, l, τ *and* ρ.
(2) Apppy the tests for theses values for the three cases.
(3) Take the domain which gives the greater p-value upon it is greater that 5%.

5. CONCLUSION

The remainding case 0$\leq \tau \leq 1/2$ is to be studied later with more restrictive conditions. The continuous asymptotic convergence of theses processes in C([T,T']) spaces, $1/2 < T < T\prime$, will be studied in future research works. An other research topic is the refinement of the theorems by replacing the centering constant in (2.16) by zero, $1/(\tau\gamma)$ or $(\mathrm{y}_0 - G^{-1}(1-k/n))/(\gamma+1)$ according to F$\in D(\Lambda)$, F$\in D(\varphi_\gamma)$ or F$\in D(\psi_\gamma)$ under the second order conditions of de Haan and Stadtmüller [5] and in (2.17) by $(\gamma\rho+1)/(\gamma\tau+1)$.

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